AMERICAN

Journal of Mathematics

EDITED BY

FRANK MORLEY

WITH THE COÖPERATION OF

A. COHEN, CHARLOTTE A. SCOTT, A. B. COBLE

AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

Πραγμάτων ἔλεγχος οὐ βλεπομένων

VOLUME XLII

BALTIMORE : THE JOHNS HOPKINS PRESS

LEMCKE & BUECHNER, *New York*
E. STEIGER & CO., *New York*
G. E. STECHERT & CO., *New York*
WILLIAM WESLEY & SON, *London*
A. HERMANN, *Paris*
ARTHUR F. BIRD, *London*

1920

P16737

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

INDEX

Groups of Order $2^{m'}$ Which Contain a Relatively Large Number of Operators of Order Two.

By G. A. Miller.

§ 1. *Introduction*

When exactly one-half of the operators of a group are of order 2 the group is obtained by extending an abelian group of odd order by means of an operator of order 2 which transforms each operator of this abelian group into its inverse, and if a group is constructed in this manner then exactly one-half of its operators are of order 2. If more than one-half of the operators of a group are of order 2 this excess must always be a number of the form $2^a - 1$ and there is an infinite system of such groups for every positive integral value of a.*

From the fact that the number which expresses the excess of the operators of order 2 over half the order of the group is always of the form $2^a - 1$ it results directly that in a group of order $2^{m'}$ the number of operators whose orders exceed 2 is always equal to the order of the group multiplied by a number of the form $\frac{2^n - 1}{2^{n+1}}$ and there are groups in which this product gives the number of the operators whose orders exceed 2 for every positive integral value of n. The smallest relative number of operators whose order exceeds 2 in a group of order $2^{m'}$ is therefore one-fourth of the order of the group in case there is at least one such operator.

Let G be any group in which more than one-half of the operators are of order 2. When G is abelian it is evidently of type $(1, 1, 1, \cdots)$ and hence we shall assume in what follows that G is non-abelian. As more than one-half of the operators of G are of order 2 these operators generate G and hence some of the operators of order 2 contained G must be non-invariant. If we let H_1 represent the subgroup composed of all the operators of G which are commutative with one such non-invariant operator s_1 then more than one-half of the operators of H_1 must be of order 2.

If H_1 is non-abelian it contains a non-invariant operator s_2 of order 2, and we may represent by H_2 the subgroup of H_1 composed of all its operators which are commutative with s_2. The subgroup H_1 includes the central of G

* *Bulletin of the American Mathematical Society*, Vol. XXV (1919), p. 33).

and H_2 includes this central as well as s_1. If H_2 is non-abelian we can continue this process until we arrive at an abelian group H_m which includes the central of G and $s_1, s_2, \cdots, s_{m-1}$. The group H_m is evidently of type $1, 1, 1, \cdots$). In the article to which we referred it was proved that $m = 1$ whenever the order of G is divisible by some odd prime number but it is possible to construct groups of order $2^{m'}$ in which m exceeds any given integer.

All the operators of G can be uniquely represented in the following form

$$H_m + H_m t_2 + H_m t_3 + \cdots + H_m t_r,$$

where it may be assumed without loss of generality that each of the operators $t_2, t_3, \cdots, t_r$ is of order 2. It is known that whenever the order of G is divisible by an odd prime number then exactly one-half of the operators in each of the co-sets $H_m t_a$, $2 \leqq a \leqq r$, are of order 2 and that in no case can more than one of these co-sets involve more operators whose orders exceed 2 than operators of order 2. In the following section we shall prove that when there is one such co-set and $r > 2$ then exactly one-fourth of its operators are of order two.

§ 2. *Less than one-half of the operators in one co-set are of order two*

The co-set in which less than one-half of the operators are of order 2 is characterized by the fact that it is composed of all the operators of G which are commutative with less than one-half of the operators of H_m. In particular, H_m contains an operator s_1 which transforms an operator t_1 of this co-set into its inverse and hence if H_1 is defined as above it does not include any operator of this co-set. Every operator of order 2 found in H_1 but not in H_m $r > 2$, is either commutative with t_1 or transforms t_1 into its inverse, for if the product of t_1 and this operator of order 2 has an order which exceeds 2 then this product is transformed into its inverse by s_1 and hence the said operator of order 2 and t_1 are commutative.

The operators of order 2 contained in H_1 but not in H_m must either generate H_1 or an abelian subgroup of type $(1, 1, 1, \cdots)$ and index 2 under H_1. The former of these alternatives is impossible since t_1 cannot be commutative with one-half of the operators of H_m. Otherwise one-half of the operators of the co-set containing t_1 would be of order 2. It therefore results that when $r > 2$ exactly one-fourth of the operators of H_m are commutative with t_1 and that the central of H_1 is a subgroup of index 2 under H_m. As exactly one-fourth of the operators of the co-set containing t_1 are of order 2 we have proved the following theorem. *Whenever one of the co-sets*

$H_mt_2, \cdots, H_mt_r, r > 2$, contains more operators of order exceeding 2 than of order 2 then exactly one-fourth of the operators of this co-set are of order 2.

It may be assumed without loss of generality that H_mt_r is the co-set in which less than one-half of the operators are of order 2, and that H_1 is composed of the co-sets $H_mt_2, \cdots, H_mt_\delta$ in addition to H_m. When $\delta = 2$ the subgroup H_m is invariant under H_1 because it is of index 2. If $\delta > 2$ this subgroup is still invariant since it is generated by all the operators of order 2 found in H_1 and having more than two conjugates under H_1. It has therefore been proved that H_m is invariant under H_1 independently of the value of δ.

It is now easy to prove that H_m is also invariant under G. In fact, the operator s_1 could have been so selected that it would be commutative with all the operators of any one of the given co-sets except the co-set H_mt_r, and hence H_m is invariant under each of these co-sets. It therefore results that *whenever more than one-half of the operators in one of the co-sets H_mt_a, $2 \leqq a \leqq r$ have orders which exceed 2 then H_m is invariant under G.*

The quotient group G/H_m involves only operators of order 2 besides the identity and hence this quotient group must be abelian and of type $(1, 1, 1, \cdots)$. The operators of H_m which are commutative with t_r constitute the central of G and include the commutators of G which arise from operators in H_m. The former of the theorems involved in the last sentence results directly from the fact that these operators are also commutative with every operator in the co-sets H_mt_a, $2 \leqq a \leqq \delta$, while the latter is a consequence of the fact that each of the operators $t_2, \cdots, t_r$ transforms H_m into an automorphism of order 2. If the commutator of order 2 arising from t_a, $2 \leqq a \leqq r-1$, would not be commutative with t_r then t_at_r could not transform H_m according to an automorphism of order 2.

Since all the commutators arising from operators of H_m are found in the central of G and since each of the operators $t_2, \cdots t_{r-1}$ is commutative with exactly one-half the operators of H_m it results that all the commutators with respect to H_m arising from these operators must also arise from t_r. As only three such commutators of order 2 arise from t_r it results that $\delta = 2$ and $r = 4$. The central of G may have any order of the form $2^{m''}$, $m'' > 1$ and there is evidently one and only one such group for every possible value of m''. Hence the following theorem has been established. *There is one and only one group of order 2^a, $a > 5$, such that one of the co-sets with respect to H_m involves more operators whose orders exceed 2 than operators of order 2. This group contains an abelian subgroup of index 4 and of type $(1, 1, 1, \cdots)$ and a central of index 16. The central quotient group is abelian and of type $(1, 1, 1, \cdots)$, and the commutator subgroup is of order 4.*

When $a > 6$ this group is evidently the direct product of the group of order 64 which belongs to this system and the abelian group of order 2^{a-6} and of type $(1, 1, 1, \cdots)$. In the form in which G presented itself above $m = 2$ but it is possible to select for s_1 an operator of H_m so as to make $m = 1$. If this is done the order of H_m remains the same since one of the corresponding co-sets must have less than one-half of its of order 2. When half of the operators of each co-set must be of order 2 it is clear that the order of H_m is independent of the choice of the set of operators $s_1, s_2, \cdots s_m$. Hence it results from the property of the groups just considered that *the order of H_m is an invariant of every group in which at least one-half of the operators are of order 2 and $r > 2$.*

It should also be noted that when $r > 2$ and G has a co-set, with respect to H_m, in which less than one-half the operators are of order 2 it must have such a co-set with respect to every possible H_m. Hence the possession of such a co-set is an invariant property of the group. When $r = 2$ this is clearly not the case, but when this condition is satisfied m is an invariant, being equal to unity, and H_1 is one of two groups. One of these must be of index 2 while for the other r may exceed 2.

If both of these subgroups are of index 2 exactly one-fourth of the operators of G have an order greater than 2 and G is known to be the direct product of the octic group and an abelian group of order 2^a and of type $(1, 1, 1, \cdots)$. When one of them has an index greater than 2 then each of the co-sets which arise with respect to this subgroup has exactly half of its operators of order 2 while the remaining operators of this co-set are of order 4 and have a common square. The co-set with respect to the other of these two subgroups contains more operators of order 4 than of order 2, the number of the operators of order 2 being half the order of the smaller subgroup which may be used for H_1. The least order which this subgroup can have when m' is odd is $\frac{1}{2}(m' - 1)$. When m' is even it is $\frac{1}{2}(m' - 2)$ since the commutator subgroup appears in the central of G.

In this system of groups for which r may be equal to 2 it is evident that there is an infinite number of groups in which the number of operators is equal to the order of the group multiplied by a number of the form $\frac{2^n - 1}{2^{n+1}}$, where n is an arbitrary positive integer. In fact, if one such group is found we can find an infinitude of others by forming the direct product of this group and abelian groups of order 2^a and of type $(1, 1, 1, \cdots)$, where a represents a positive integer but is not otherwise limited.

§ 3. *Exactly one-half of the operators of each co-set are of order 2 and $m > 1$.*

When $m > 1$ and exactly one-half of the operators of each co-set are of order 2 it is easy to see that it is not possible that the subgroup of index 2 under H_m composed of all its operators which are commutative with each of the operators of one co-set is the same for all the different co-sets. Let t be any operator of order greater than 2 contained in G and let s_1 be any operator of H_m which is commutative with t but not contained in the central of G. Let H_1, as before, represent the subgroup composed of all the operators of G which are commutative with s_1.

The product obtained by multiplying any operator of order 2 contained in H_1 into an arbitrary operator whose order exceeds 2 in $G - H_1$ is transformed into its inverse by s_1 whenever the order of this product exceeds 2. In this case the two factors of this product must be commutative. As every operator whose order exceeds 2 in $G - H_1$ is transformed either into itself or into its inverse by every operator of order 2 in H_1 it results that all the operators of H_1 transform every operator of $G - H_1$ whose order exceeds 2 either into itself or into its inverse. The squares of each of the operators of H_1 is therefore contained in the central of G. In particular, t^2 is in this central. As t was any operator of order greater than 2 contained in G it results that *the squares of all the operators of G are found in the central of G and hence G involves no operator whose order exceeds* 4.

Since the quotient group of G with respect to its central involves only operators of order 2 besides the identity this quotient group must be abelian. In particular, both H_1 and H_m must be invariant subgroups of G. Every non-invariant operator of order 2 contained in G has just two conjugates. For, if such an operator s_1 had more than two conjugates, it may be supposed to appear in H_m. It may be assumed that s_1 is transformed into itself multiplied by the three distinct operators s_0, s'_0, $s_0 s'_0$ of order 2 contained in the central of G. Let t_0, t'_0, $t_0t'_0$ be three operators of G which effect these transformations respectively. If t_0 and t'_0 were not commutative with the operators of the same subgroup of index 2 under H_m there would be a co-set corresponding to H_m which would have less than half of its operators of order 2.

Hence it may be assumed that t_0 and t'_0 are commutative with the operators of the same subgroup of index 2 under G. There must be an operator t''_0 in G which is not commutative with all the operators of this subgroup. The operators of H_m which are commutative with both of the operators t_0 and t''_0 constitute a subgroup of index 4 under H_m and include the central of G. Hence t_0 and t''_0 are non-commutative with the operators of a common co-set

with respect to this subgroup and it may be assumed without loss of generality that they transform the operators of this co-set into themselves multiplied by two distinct operators since t_0 may be replaced by t'_0. Hence we are again led to a co-set with respect to H_m which involves more operators whose orders exceed 2 than operators of order 2. It has therefore been proved that when $m > 1$ for every possible choice of s_1, and one-half of the operators of each co-set with respect to H_m are of order 2 then each non-invariant operator of order 2 contained in G has exactly two conjugates under G.

From the fact that each non-invariant operator of order 2 contained in G has exactly two conjugates under G it is easy to determine additional fundamental properties of G. In particular, it is possible to select a set of independent generators of G in such a way that all of these generating operators are of order 2 and that each one is commutative with all of the others with the exception of at most one of them. In fact, we may select s_1 as the first independent generator of G and then let s_2 be any operator of order two in $G - H_1$. The operators common to the two subgroups composed of all the operators of G which are commutative with s_1 and s_2 respectively constitute a subgroup of index 4 under G which includes the central of G. The order of the product of s_1 and s_2 is 4.

If this subgroup of index 4 is abelian it must be of type $(1, 1, 1, \cdots)$ and H_1 must be abelian and of the same type. In this case m would be equal to 1 and hence we may assume that the subgroup in question is non-abelian. More than one-half of its operators must be of order 2 since more than one-half of the operators of G are of order 2. It is therefore possible to find two additional non-commutative generators of order 2, s_3 and s_4, such that each of these generators is commutative with both of the operators s_1, s_2 and that $(s_3 s_4)^2 = (s_1 s_2)^2$. If the last condition were not satisfied $s_1 s_3$ would be an operator of order 2 having more than two conjugates under G.

When the group composed of all the operators of G which are commutative with each of the four operators s_1, s_2, s_3, s_4 is abelian and of order greater than 2 it is of type $(1, 1, 1, \cdots)$ and G is the direct product of a subgroup of index 2 under this abelian group and the group of order 32 generated by the given four operators. On the other hand, when the operators of G which are commutative with each of the four operators s_1, s_2, s_3, s_4 are not all commutative with each other then the subgroup generated by these operators must involve two non-commutative operators of order 2, s_5 and s_6, such that the square of their product is equal to $(s_1 s_2)^2$.

This process can evidently be continued until we obtain either a set of generators of G or an invariant subgroup of G of order 2^{2a+1} such that G is

the direct product of this subgroup and an abelian group of type $(1, 1, 1, \cdots)$. Hence we have established the following theorem: *When m exceeds unity for every possible choice of H_m and one-half of the operators in each co-set of G with respect to H_m are of order 2 then G is generated by β operators of order 2 such that each of them is commutative with all of the others except at most one of them and the square of the product of every pair of non-commutative operators of this set is the commutator of order 2 contained in G.*

§ 4. *Exactly one-half of the operators of each co-set are of order 2 and $m = 1$*

Two cases require consideration according as $m = 1$ for every possible choice of operators from G for s_1 or as $m = 1$ for some one but not for every possible such choice. We shall first restrict our attention to the former of these cases and hence it results that the subgroup of index 2 under H_1 composed of all the operators of H_1 which are commutative with the operators of one of the $r - 1 > 1$ co-sets constitute the central of G.

Let t_1 and t_2 be two operators of order 2 found in any two distinct co-sets with respect to H_1. If t_1 and t_2 were commutative the central of G together with t_1 and t_2 would generate an abelian group of type $(1, 1, 1, \cdots)$ and of a larger order than the order of H_1. By using t_1 instead of s_1 we would therefore find a new group for H_1 which would give rise to a co-set in which less than one-half of the operators are of order 2. As this case has been considered it may be assumed that t_1 is not commutative with any operators of order 2 except those found in the central of G or in the co-set to which t_1 belongs.

Since t_1 transforms into its inverse each of the products obtained by multiplying t_1 successively into all the operators of order 2 contained in G, and as these products include all the operators of G whose orders exceed 2 it results that every operator G whose order exceeds 2 is transformed only into itself or into its inverse under G and that it must be transformed into itself by all the operators of G whose order exceeds 2 since it is transformed into its inverse by a number of operators of order 2 equal to one-half of the order of G. Hence it results that the operators whose orders exceed 2 in G generate an abelian group and that *G is either the dihedral or the extended dihedral group.*

It remains to consider the case when $r > 2$ and H_1 involves an operator s_1 which transforms every operator whose order exceeds 2 in G into its inverse and when, moreover, G contains a non-invariant operator s_2 which is commutative with operators whose orders exceed 2 in G. If the central of G is composed of half of the operators of H_1 then s_2 must be contained in an

abelian group of type $(1, 1, 1, \cdots)$ whose order exceeds that of H_1. As one of the co-sets of G with respect to this abelian group would contain more operators whose orders exceed 2 than operators of order 2 it may be assumed that the order of the central of G is less than one-half of the order of H_1. It therefore results that the square of every operator of G is found in the central of G and hence H_1 is an invariant subgroup of G.

We may form H_m, $m > 1$, by starting with s_2 instead of with s_1. Let H'_2 be the subgroup of G composed of all its operators which are commutative with s_2. The product of an operator t_1 of order 4 in H'_2 and an operator t_2 of order 4 in $G - H'_2$ is transformed into its inverse by each of the operators s_1 and s_2 whenever the order of this product exceeds 2. In this case, it results therefore from a transformation by s_1 that $t_1^{-1}\, t_2^{-1} = t_2^{-1}\, t_1^{-1}$ and hence t_1 and t_2 are commutative. On the other hand, it results from a transformation by s_2 that $t_1 t_2^{-1} = t_1^{-1} t_2^{-1}$. As the latter is impossible it follows that the product of an operator of order 4 in H'_2 and an operator of this order in $G - H'_2$ must be of order 2.

As t_2 is commutative with one-half of the operators of H_m and is transformed into its inverse by the other half it results that it is commutative with all the operators of order 2 in $H'_2 - H_m$. Since t_2 is any operator of order 4 in $G - H'_2$ all of these operators are commutative with the operators of the same subgroup of index 2 under H_m. This is contrary to the hypothesis that s_2 is commutative with t_1 but not with t_2. Hence it results that *if exactly half the operators in each co-set with respect to every possible H_m are of order 2 and $m = 1$ for one choice of s_1 it must be equal to unity for every such possible choice.* Hence the central of G is of index 2 under H_1.

§ 5. *Conclusion*

The most familiar system of groups in which at least one-half of the operators are of order 2 are the dihedral and the generalized dihedral groups and when the order of a group is divisible by an odd prime number there are no other groups which have the property that at least one-half of their operators are of order 2. When the order of G is $2^{m'}$ there are other infinite systems of groups such that the number of operators of order 2 in the group exceeds half the order of the group but in all these groups the order of each operator is a divisor of 4. The dihedral and the generalized dihedral groups may be characterized by the facts that at least one-half of their operators are of order 2, that all the operators of such a group which are commutative with an arbitrary non-invariant operator of order 2 constitute an abelian group of

type $(1, 1, 1, \cdots)$ and that they do not contain an abelian group of this type which is of even order and of index 2 unless the central is of index 4 or of index 1.

One of the simplest systems of groups which are neither dihedral nor extended dihedral but have the property that more than one-half of their operators are of order 2 is obtained by extending the abelian group of order $2^{m'-1}$ and of type $(1, 1, 1, \cdots)$ by means of an operator of order 2 which transforms this abelian group into itself but is commutative with less than one-half of its operators. The order of the central of such a group cannot be less than $\frac{1}{2}(m'-1)$ since the multiplying group must appear in the central. Hence these groups are possible when and only when $m' > 4$ and the number of such groups for a given value of m' is $\frac{1}{2}(m'-1)-1$ when m' is odd and $\frac{1}{2}(m'-2)-1$ when m' is even. It may be noted that for each of these groups $m=1$, but that r has two possible values.

As a second system composed of groups which are neither dihedral nor extended dihedral but have the property that more than one-half of their operators are of order 2 we may consider those for which one co-set with respect to every possible H_m has fewer operators of order 2 than of larger order. The smallest group of this system is of order 64 and every other group of the system is the direct product of this group of order 64 and an abelian group of order 2^a and of type $(1, 1, 1, \cdots)$. Exactly 7/16 of the operators of each of these groups are of order 4.

Finally, there is a third system of groups which are neither dihedral nor extended dihedral but contain more operators of order 2 than operators of larger order. This system is characterized by the facts that half of the operators in each co-set with respect to H_m are of order 2 and $m > 1$. The smallest order of such a group is 2^5 since if it is not dihedral or generalized dihedral it must contain at least two pairs of non-commutative generators when it is generated by a set of $\beta \gtreqless 4$ operators of order 2 such that each operator of this set is commutative with all except at most one other operator of the set. Every non-invariant operator of such a group has two conjugates under the group. By adding to the three systems just described the system composed of the dihedral and the extended dihedral groups we obtain the four possible systems of groups of order $2^{m'}$ which involve all the groups whose order is of this form and which have the property that more than one-half of their operators are of order 2.

To exhibit an important analogy between the second and the third system described above it may be desirable to note that the second system may be characterized by the fact that each of its groups may be generated by a set of

$\beta \geqq 4$ operators of order 2 such that each of these operators is commutative with all of the others except at most one of them and that there are exactly two pairs of non-commutative operators in the set, whose products are of order 4 and have different squares. This system can also be defined by the fact that each of its groups contains the direct product of two octic groups, and when its order exceeds 64 it is the direct product of this group of order 64 and an abelian group of type $(1, 1, 1, \cdots)$.

Case II. If $c = 2z_0$ or $c - z_0 = z_0$, then z' would be real only when $z = z_0$ and would therefore be identically zero. In this case the particle would revolve in the horizontal plane $z = z_0$, the centrifugal force and the reaction of the surface being just sufficient to overcome gravity.

Case III. If $c > 2z_0$ or $c - z_0 > z_0$ it can be shown by an argument similar to that used in discussing Case I that the particle oscillates between the same two planes $z = z_0$ and $z = c - z_0$, as in Case I; but the latter plane is the higher plane.

Thus if c is positive and not less than z_0, the vertical motion of the particle is periodic, and therefore the last equation of (8) must admit a periodic solution.

It could also be shown by the usual analytic existence proof that the last equation of (8) admits a periodic solution. It is not necessary, however, to establish the existence of a periodic solution by either method for, by Macmillan's theorem, quoted below, it is shown that if the formal construction of a periodic solution can be made, then this solution will converge under suitable conditions. Macmillan's theorem is as follows: *

If

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n;\ \mu, t) \quad (i = 1, \cdots, n) \tag{11}$$

is a system of differential equations in which the f_i are expansible as power series in $x_1, \cdots, x_n$, and μ, vanishing for $x_1 = \cdots = x_n = \mu = 0$, with coefficients which are uniform, continuous, and periodic functions of t with the period $2j\pi$: and if the f_i converge for $0 \leqq t \leqq 2j\pi$, when $|x_j| < \rho_j$, $|\mu| < r$, then the solutions $x_i(t)$ are expansible as power series in μ, or any fractional power of μ, which converge for all t in the interval $0 \leqq t \leqq 2j\pi$ provided $|\mu|$ is sufficiently small. If the constants of integration can be determined at each step so as to make the solution formally periodic with the period $2j\pi$, then the solutions so determined will be periodic and converge for all finite values of t provided $|\mu|$ is sufficiently small.

§ 4. *The Vertical Motion in terms of Elliptic Integrals.* The integral (10) may be written

$$\int \frac{\sqrt{p + 2z}\ dz}{\sqrt{(z_0 - z)\ (z - c + z_0)}} = 2\sqrt{g} \int_{t_0}^{t} dt = 2\sqrt{g}(t - t_0). \tag{10'}$$

We desire to express this integral in terms of elliptic integrals. The neces-

* *Trans. Am. Math. Soc.*, Vol. XIII, No. 2, pp. 146-158.

where λ_0 is defined in (6). The last equation of (8) is independent of the first two and admits the integral

$$z'^2 = \frac{4gz(c-z) + c_1}{p + 2z}, \tag{9}$$

where c_1 is the constant of integration.

If the vertical motion is to be periodic, z cannot increase indefinitely and therefore z' must vanish for some value of t, $t = t_0$, say. Suppose $z = z_0$ at $t = t_0$. Then since $z' = 0$ at $t = t_0$, it follows that the constant of integration in (9) has the value

$$c_1 = -4gz_0(c - z_0).$$

Hence the integral (9) becomes

$$z' = \pm\sqrt{\frac{4g}{p+2z}(z_0 - z)(z_0 - c + z)}. \tag{10}$$

It is readily seen from (10) that z' vanishes for $z = z_0$, and also for $z = c - z_0$.

As p is assumed to be positive, no part of the paraboloid (7) will lie below the xy-plane. Then z and z_0 cannot be negative and therefore, for real initial conditions, c cannot be less than z_0. Three cases arise according to the values assigned to c. They are:

Case I. $z_0 \leqq c < 2z_0$.
Case II. $c = 2z_0$.
Case III. $c > 2z_0$.

Case I. Suppose $z_0 \leqq c < 2z_0$ or $c - z_0 < z_0$. Then z' is real so long as z lies in the interval $z_0 \gtreqless z \gtreqless c - z_0$. If the particle is started in the plane $z = z_0$, it cannot remain in this plane for $z = z_0 \neq 0$ does not satisfy the z-equation of (8). Moreover, it cannot move above this plane since $z < z_0$, hence it must move below this plane. Consequently z decreases or z' is negative, and the negative sign must be taken in (10). The particle continues to fall until it reaches the plane $z = c - z_0$ where the velocity again vanishes. As $z = c - z_0$ is not a solution of the z-equation in (8) and as z cannot be less than $c - z_0$ it must increase, or the positive sign must be taken in (10). The particle then rises until the plane $z = z_0$ is reached where the velocity again vanishes and the radical in (10) changes sign. Hence the particle oscillates between the two horizontal planes $z = z_0$ and $z = c - z_0$, the latter plane being the lower plane.

§ 2. *The Differential Equations.* If the particle is of unit mass and moves without friction, then the differential equations of motion are

$$x'' = X, \quad y'' = Y, \quad z'' = Z - g,$$

where X, Y, and Z are the normal reactions due to the surface, and g is the acceleration due to gravity. Since the surface is assumed to be smooth, the normal reactions at any point are proportional to the direction cosines of the normal at that point, and therefore the differential equations become

$$\left.\begin{aligned} x'' &= X = \lambda F_x = 2\lambda x, \\ y'' &= Y = \lambda F_y = 2\lambda y, \\ z'' &= Z - g = \lambda F_z - g = 2\lambda(-p + \epsilon f_z) - g, \end{aligned}\right\} \tag{3}$$

where λ is a factor of proportionality.

These equations admit the vis viva integral

$$x'^2 + y'^2 + z'^2 = 2g(c - z), \tag{4}$$

where c is the constant of integration.

Since the equation of constraint (2) does not contain t explicitly, the factor λ can be obtained by differentiating $F(x, y, z)$ twice with respect to t and eliminating x'', y'', and z'' from the result by means of the differential equations (3). Then on making use of the relations

$$x^2 + y^2 = 2pz - 2\epsilon f, \quad x'^2 + y'^2 = -z'^2 + 2g(c - z),$$

from (2) and (4), respectively, we find

$$2\lambda = \frac{2g(z - c) - g(p - \epsilon f_z) + z'^2(1 - \epsilon f_{zz})}{p^2 + 2pz - 2\epsilon(f + pf_z) + \epsilon^2 f_z^2}. \tag{5}$$

The part of 2λ which is independent of ϵ is

$$\lambda_0 = \frac{2g(z - c) - gp + z'^2}{p(p + 2z)}. \tag{6}$$

(A). PERIODIC ORBITS ON A PARABOLOID OF REVOLUTION.

§3. *Proof of Existence of Periodic Orbits.* For $\epsilon = 0$ the surface (2) becomes

$$x^2 + y^2 - 2pz = 0, \tag{7}$$

and the differential equations (3) become

$$x'' = \lambda_0 x, \quad y'' = \lambda_0 y, \quad z'' = -p\lambda_0 - g, \tag{8}$$

Periodic Orbits on a Surface of Revolution.*

By Daniel Buchanan.

§ 1. *Introduction.* The object of this paper is to determine the periodic orbits described by a particle which moves, subject to gravity, on a smooth surface of revolution, the axis of which is vertical. Let us denote the equation of the surface by

$$x^2+y^2-2pz+2\epsilon\phi\,(x^2+y^2,\,z)=0, \tag{1}$$

where p is a positive constant, ϵ an arbitrary parameter, and ϕ a power series in x^2+y^2 and z, converging for x^2+y^2 and $|z|$ sufficiently small. There will be no loss of generality if we suppose that the constant term of ϕ is zero, otherwise it could be eliminated by a linear substitution for z, that is, by a translation of the xy-plane along the z-axis. Since the equation (1) vanishes with x^2+y^2 and z, it may be solved, by the theory of implicit functions, for x^2+y^2 as a power series in z converging for $|z|$ sufficiently small. Hence the equation (1) may be expressed as

$$F(x,y,z)=x^2+y^2-2pz+2\epsilon f(z)=0, \tag{2}$$

where f is a power series in z converging for $|z|$ sufficiently small.

For $\epsilon=0$ the equation (2) represents a vertical paraboloid of revolution. The generating parabola has its axis coinciding with the positive z-axis and its semi-latus-rectum equal to p. Periodic orbits are first constructed when the particle moves on this paraboloid. Then the analytic continuation of these orbits is made with respect to ϵ and in this way the orbits for the more general surface (2) are determined. It was for this reason that the parameter ϵ was introduced in (2).

The form of the equation of the surface (2) was suggested by a somewhat similar equation used by Poincaré in his memoir, *Sur les Lignes Géodésiques des Surfaces Convexes: Trans. Am. Math. Soc.*, vol. vi (July, 1905). Geodesic lines play the same rôle in Poincaré's memoir as periodic orbits do in this paper.

It will be readily seen that when $\epsilon=0$ the problem here considered is somewhat analogous to the well-known problem of the spherical pendulum.†

* Presented to the American Mathematical Society, Sept. 5, 1918.

† For a complete discussion of the spherical pendulum, including the horizontal as well as the vertical motion, see Moulton's *Periodic Orbits*, chap. III, also *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXII (1911), pp. 338-365.

VIII

Solutions when there are equalities and congruences among the $a_j^{(0)}$.

Suppose that two of the $a_j^{(0)}$, for example $a_1^{(0)}$ and $a_2^{(0)}$ are equal, and that a third one, say $a_3^{(0)}$, differs from $a_1^{(0)}$ by an imaginary integer. Furthermore we shall assume that there are no other equalities or congruences among the $a_j^{(0)}$. Two cases arise here: (a) the solutions are of the form (54) with $a_1^{(0)} = a_2^{(0)}$; (b) the solutions are of the form (55)

Case (a). In this case we have

$$D_0 = | c_{ij} | [1-\sigma(1-e^{2a_1^{(0)}\pi})]^3 \prod_{j=4}^{\infty} [1-\sigma(1-e^{2a_j^{(0)}\pi})].$$

In setting $a = a_1^{(0)} + \delta_1$, we find that the term in D of lowest degree in δ_1 alone is of the third. To get the term in D of lowest degree in μ alone we set $\delta_1 = 0$. Then it becomes evident at once that each of the first columns carry μ as a factor, while the remaining ones do not. Consequently the term of lowest degree in μ is of the third degree at least. Furthermore since the first three columns vanish when $\mu = \delta_1 = 0$ there are no terms of lower degree than the third in δ_1 and μ. Hence in general we see D of the form

$$(60) \qquad D = \delta_1^3 + \gamma_{21}\delta_1^2\mu + \gamma_{03}\mu^3 + \cdots = 0.$$

The problem is now one of implicit functions. The details of the special cases must be treated as they arise. However, we make the general statement that since the roots of the cubic terms of (60) set equal to zero are in general distinct, it follows from the theory of implicit functions that the three values of δ_1 are in general expansible in integral powers of μ.

Case (b). In this case we have

$$D_0 = [1-\sigma(1-e^{2a_1^{(0)}\pi})]^3 \prod_{j=4}^{\infty} [1-\sigma(1-e^{2a_j^{(0)}\pi}) = 0.$$

On introducing δ_1 as before, we find that the term of lowest degree in δ_1 alone is of the third degree. But when the terms involving μ are retained in D, only the first and third columns vanish when $\mu = \delta_1 = 0$, and consequently the expansion of D will contain a term in μ^2 alone. Furthermore, since the first and third columns vanish for $\mu = \delta_1 = 0$, there will be no terms of degree lower than the second in μ and δ_1. Hence in general D has the form

$$(61) \qquad D = \delta_1^3 + \gamma_{11}\delta_1\mu + \gamma_{02}\mu^2 + \cdots = 0.$$

In the general case in which γ_{11} and γ_{02} are not zero, there is one solution in integral powers of μ and two in powers of $\sqrt{\mu}$.

When the roots $a_j^{(0)}$ have higher multiplicities and more congruences among them, we make a similar discussion.

in infinitely many variables in constant coefficients and from IV, the solutions are of the form

$$(54)\qquad x_i = \sum_{j=1}^{\infty} A_j[c_{ij}e^{a_j^{(0)}t} + \sum_{k=1}^{\infty} x_{ij}^{(k)}\mu^k] \quad \text{or}$$

$$(55)\qquad x_i = A_1[c_{i1}e^{a_1^{(0)}t} + \sum_{k=1}^{\infty} x_{i1}^{(k)}\mu^k] + A_2[(c_{i2} + tc_{i1})e^{a_1^{(0)}t} + \sum_{k=1}^{\infty} x_{i2}^{(k)}\mu^k] + \sum_{j=1}^{\infty} A_j[c_{ij}e^{a_j^{(0)}t} + \sum_{k=1}^{\infty} x_{ij}^{(k)}\mu^k].$$

After setting $e^{-2a\pi} = \sigma/\sigma - 1$, the fundamental equation becomes either

$$(56)\quad D(\sigma,\mu) = |\,[\sigma(c_{ij}e^{2a_j^{(0)}\pi} + \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi)\mu^k) - c_{ij}(\sigma - 1)]\,| = 0 \quad \text{or}$$

$$(57)\quad D(\sigma,\mu) = |\,[\sigma\{c_{i1}e^{2a_1^{(0)}\pi} + \sum_{k=1}^{\infty} x_{i1}^{(k)}(2\pi)\mu^k\} - c_{i1}(\sigma - 1)],$$
$$[\sigma\{(c_{i2} + 2\pi c_{i1})e^{2a_1^{(0)}\pi} + \sum_{k=1}^{\infty} x_{i2}^{(k)}(2\pi)\mu^k\} - c_{i2}(\sigma - 1)], \cdots | = 0.$$

For $\mu = 0$ both of these, by the theory of infinite determinants, reduce to

$$(58)\quad D_0 = |\,c_{ij}\,|\,[1 - \sigma(1 - e^{2a_1^{(0)}\pi})]^2 \prod_{j=3}^{\infty} [1 - \sigma(1 - e^{2a_j^{(0)}\pi})].$$

As before we set $a = a_1^{(0)} + \delta_1$, and expand as a power series in δ_1. Then we see that the term of lowest degree in δ_1 alone is of the second. When the determinant is of the form (56) with $a_1^{(0)} = a_2^{(0)}$, the term of lowest degree in μ alone is of the second in general. Then we have the same form as in VI. But if the determinant is of the form (57), the term in μ alone is in general of the first degree. In the former case we have a consideration similar to that in VI; in the latter case, in general the solutions for δ_1 are of the form

$$(59)\qquad \begin{aligned} \delta_{11} &= \mu^{1/2}P(\mu^{1/2}) \\ \delta_{12} &= \mu^{1/2}P(-\mu^{1/2}), \end{aligned}$$

where P is a power series in $\sqrt{\mu}$ and contains a term independent of μ. The discussion of the special cases is made just as in VI. On substituting these expansions for $a = a_1^{(0)} + \delta_1$ in (44) we solve for the A_j as power series in $\sqrt{\mu}$. These A_j substituted in (42) give y_{i1} and y_{i2} as power series in $\sqrt{\mu}$.

If p of the $a_j^{(0)}$ are equal, then for these roots the expansions of D starts with δ_1^p as the term of lowest degree in δ_1 alone, and except in the special cases corresponding to those mentioned in the foregoing, the term in μ alone is of the first degree. Consequently in general for $a_1^{(0)} = a_2^{(0)} \cdots = a_p^{(0)}$, we have

$$\delta_{ij} = \epsilon^j\mu^{1/p}P(\epsilon^j\mu^{1/p}),\ j = 0, 1, \cdots, p - 1,$$

where ϵ is a pth root of unity.

Then since a_1 is not a multiple zero of D, not all the first minors of D are zero when $a = a_1$. The ratios of the A_j will be determined from (44). If $\mu\Delta$ is a non-vanishing first minor corresponding to an element in the first column of D, it follows from the form of (45), remembering that we have $x_{ij}^{(0)}(t) = c_{ij} e^{a_j^{(0)} t}$, that solving (44), we get

$$A_2 = \frac{\mu\Delta_2}{\mu\Delta} A_1, \quad A_j = \frac{\mu^2\Delta_j}{\mu\Delta} A_1, \quad j = 3, 4, \cdots, \infty,$$

where $\Delta = \Delta^{(0)} + \Delta^{(1/2)}\mu^{1/2} + \Delta^{(1)}\mu + \cdots\cdots,$

$$\Delta_j = \Delta_j^{(0)} + \Delta_j^{(1/2)}\mu^{1/2} + \Delta_j^{(1)}\mu + \cdots\cdots.$$

On substituting these series for the A_j in (42) we find that the y_{i1} are developable as series of the form

$$y_{i1} = y_{i1}^{(0)} + y_{i1}^{(1/2)}\mu^{1/2} + y_{i1}^{(1)}\mu + \cdots\cdots \qquad i = 1, 2, \cdots \infty.$$

So we see that in general the y_{i1} carry terms in $\sqrt{\mu}$ although the term in $\sqrt{\mu}$ is absent in the expansion for a_1.

However, if all the first minors corresponding to the elements of the first column are zero, and if there is a first minor distinct from zero corresponding to the elements of the second column, the results are precisely the same. But suppose that all the first minors corresponding to the elements of both the first and second columns are zero. Then suppose that a first minor corresponding to an element of the kth column is not zero. Then it follows from the form of (45) that when $a = a_j$, it will carry the factor μ^2; and let this minor be denoted by $\mu^2\Delta$. Then solving (44) we get

$$A_1 = \frac{\mu\Delta_1}{\mu^2\Delta} A_k, \quad A_2 = \frac{\mu\Delta_2}{\mu^2\Delta} A_k, \quad A_j = \frac{\mu^2\Delta_j}{\mu^2\Delta} A_k, \quad j = 3, 4, \cdots \quad \infty.$$

where Δ_1, Δ_2, Δ_j, do not in general vanish when $\mu = 0$. It follows from the first two equations that A_k must carry μ as a factor, since the A_j are finite for $\mu = 0$. Hence in this case the y_{i1} have the same form as before. Similarly the y_{i2} have the same properties.

The solutions associated with $a_3^{(0)}$, $a_4^{(0)}$, $\cdots$, are found as in the preceding case. If there are several groups of $a_j^{(0)}$ in which these congruences exist the discussion must be made for each one separately.

VII

Solutions when $a_1^{(0)}$ is a multiple root.

Now suppose that two of the $a_j^{(0)}$ are equal and only two, and that there are none of the congruences treated in VI. Let us choose the notation so that $a_1^{(0)} = a_2^{(0)}$. From our work in the theory of linear differential equations

(51) $D(\sigma, \mu) =$

$$|c_{ij}| \left[1 - \frac{1 - e^{2a_1^{(0)}\pi}}{1 - e^{2(a_1^{(0)}+\delta_1)\pi}}\right]^2 \prod_{j=3}^{\infty}\left[1 - \frac{1 - e^{2a_j^{(0)}\pi}}{1 - e^{2(a_1^{(0)}+\delta_1)\pi}}\right] + \mu F(\mu, \delta_1) = 0,$$

where as in (47) we have set $a = a_1^{(0)} + \delta_1$. The term of lowest degree in δ_1 alone is found by expanding the first bracket and turns out to be of the second degree. To get the terms in μ alone we suppress those involving δ_1, after which we get a factor μ from each of the first two columns. So we see that in general the term of lowest degree in μ alone will be in this case of the second degree. Hence we have

$$(52) \qquad D = |c_{ij}| \left[1 - \frac{1 - e^{2a_1^{(0)}\pi}}{1 - e^{2(a_1^{(0)}+\delta_1)\pi}}\right]^2 \prod_{j=3}^{\infty}\left[1 - \frac{1 - e^{2a_j^{(0)}\pi}}{1 - e^{2(a_1^{(0)}+\delta_1)\pi}}\right] + \delta_1\mu F_1(\delta_1, \mu) + \delta^2 F_2(\delta_1, \mu) = 0.$$

In a similar manner if p of the $a_j^{(0)}$ are congruent to $a_1^{(0)}$ mod $\sqrt{-1}$, then the term of lowest degree in δ_1 alone is of degree p, and in μ alone it is of at least the pth degree.

The problem of the form of the solution of (52) is one of implicit functions. Writing the first terms explicitly we have

$$\delta_1^2 + \kappa_{11}\delta_1\mu + \kappa_{02}\mu_2 + \text{terms of higher degree} = 0,$$

where $\kappa_{11}, \kappa_{02}', \cdots$, are constants independent of δ_1 and μ. On factoring the quadratic terms we get

$$(53) \qquad (\delta_1 - d_1\mu)\,(\delta_1 - d_2\mu) + \text{terms of higher degree} = 0.$$

If d_1 and d_2 are distinct, there are two solutions, and these have the form *

$$(54) \qquad \delta_{11} = d_1\mu + \mu^2 P_1(\mu), \quad \delta_{12} = d_2\mu + \mu^2 P_2(\mu),$$

where P_1 and P_2 are power series which converge for μ sufficiently small. In this case the solutions are found as in V.

But if d_1 and d_2 are equal, the character of the solution is in general quite different and depends upon the terms of higher degree than the second. In general it will be a power series in $\pm\sqrt{\mu}$. This case we shall consider in detail.

We see from the form of (53) that the expansion of a_1 as a power series in $\sqrt{\mu}$ will contain no term in $\sqrt{\mu}$ to the first power, but will have the form

$$a_1 = a_1^{(0)} + 0\mu^{1/2} + a_1^{(1)}\mu + a_1^{(3/2)}\mu^{(3/2)} + \cdots\cdots$$

Suppose that this expansion has been obtained from equation (45).

* Chrystal, "Algebra," Vol. 2, pp. 358 ff.

where $F_k(\delta_k, \mu)$ is a power series in μ and δ_k, converging for

$$|\delta_k| < \infty, |\mu| < \rho > 0.$$

Since by hypothesis no two of the $a_1^{(0)}$ differ by an imaginary integer, the expansion of (48) as a power series in δ_k and μ contains a term in δ_k of the first degree and no term independent of both μ and δ_k. Therefore we know by the theory of implicit functions that (48) can be solved uniquely for δ_k as a power series of the form

$$\delta_k = \mu P_k(\mu), \tag{49}$$

which converges for $|\mu| > 0$ but sufficiently small.

Now we substitute this value of $a = a_k^{(0)} + \delta_k$ in (44), and get an infinite number of linear homogeneous equations for the A_j whose determinant converges and is zero, but the first minors of that determinant are not all zero, since by hypothesis, the roots of $D_0 = 0$ are all distinct and no two differ by an imaginary integer. Consequently these equations determine uniquely the ratios of the A_j as power series in μ, which converge for μ sufficiently small. On substituting these ratios in (42) we have the particular solution y_{ik}, $i = 1, 2, \cdots \infty$, expanded as a power series in μ. Hence we may write it

$$y_{ik} = \sum_{j=1}^{\infty} y_{ik}^{(j)}(t)\mu^j. \tag{50}$$

Since the periodicity conditions have been satisfied,

$$y_{ik}(t + 2\pi) - y_{ik}(t) = \sum_{j=1}^{\infty}[y_{ik}^{(j)}(t + 2\pi) - y_{ik}^{(j)}(t)]\mu^j = 0,$$

for all μ sufficiently small and for all real t. Therefore

$$y_{ik}^{(j)}(t + 2\pi) - y_{ik}^{(j)}(t) = 0, \ j = 0, 1, 2, \cdots \infty,$$

whence it follows that the $y_{ik}^{(j)}$, $j = 0, 1, 2, \cdots \infty$, are separately periodic.

A solution is found in a similar fashion for each $a_j^{(0)}$.

VI

Solutions when no two of the $a_j^{(0)}$ are equal but when $a_2^{(0)} - a_1^{(0)} \equiv 0 \bmod \sqrt{-1}$.

Suppose that when $\mu = 0$ the characteristic equation has two roots such that $a_2^{(0)}$ and $a_1^{(0)}$ differ by an imaginary integer and that none of the other $a_j^{(0)}$ are congruent to $a_1^{(0)}$ mod $\sqrt{-1}$. Then we see from (45)

fore if the characteristic equation has no finite roots, then (46) has no finite roots, if the former has only a finite number of finite roots, then the latter has only a finite number of finite roots, then (46) has an infinite number of finite roots which yield independent solutions of the system (1).

Now since the solution (41) converges for all μ sufficiently small, including zero, the failure to find an a for $\mu = 0$ is not due to clumsy analytic methods, but shows that the system (1) has no solution of the form (41). From the theory of implicit functions we know that if (46) is satisfied when $a = a_0$, then we can solve for a as power series in μ, provided μ is sufficiently small. Therefore the existence of a root of the characteristic equation is both necessary and sufficient for the existence of a solution of the system (1) of the form (41), and the existence of an infinite number of roots of the characteristic equation is necessary and sufficient for the existence of a fundamental set of solutions of (1), each of the elements of which is of the form (41).

We shall study in detail the case that the characteristic equation has an infinite number of roots; and we shall use only those values of a obtained from (45) which for $\mu = 0$ reduce to the values of a obtained from the characteristic equation.

When $e^{-2a\pi} = 1$, the transformation $e^{-2a\pi} = \sigma/\sigma - 1$ can not be made. Then as in the case of constant coefficients when $a = 0$, we can make no general statement about the solution, as the determinant of the A_i diverges then.

V

Solutions when the $a_i^{(0)}$ are distinct and $a_i^{(0)} - a_j^{(0)} \not\equiv 0 \bmod \sqrt{-1}$.

The part of (45) which is independent of μ is

$$D_0 = |\, \sigma c_{ij} e^{2a_j^{(0)}\pi} - c_{ij}(\sigma - 1)\,| = |\, c_{ij}\,| \prod_{j=1}^{\infty} \left(1 + \frac{1 - e^{2a_j^{(0)}\pi}}{1 - e^{-2a\pi}}\, e^{-2a\pi}\right)$$

If (45) were an identity in μ, its roots would be the roots of (46), viz., $a = a_j^{(0)}$. Let us assume that we have the general case in which it is not an identity in μ, and set

$$(47) \qquad a = a_k^{(0)} + \delta_k.$$

Then we get

$$(48) \qquad D(\sigma, \mu) = D_0 + \mu F_k(\delta_k, \mu) =$$

$$|\, c_{ij}\,| \left(1 - \frac{e^{-2\delta_k\pi} - e^{-2(a^{(0)}_k + \delta_k)\pi}}{1 - e^{-2(a^{(0)}_k+\delta_k)\pi}}\right) \prod_{j=1}^{\infty}{}' \left(1 + \frac{1 - e^{-2a_j^{(0)}\pi}}{1 - e^{-2(a^{(0)}_k+\delta_k)\pi}}\, e^{-2(a_k^{(0)}+\delta_k)\pi}\right) + \mu F_k(\delta_k, \mu) \quad j \neq k.$$

to be determined. After making the transformation (41), the differential equations and their solutions become

$$(42)\begin{cases} y'_i + ay_i = \sum_{j=1}^{\infty}[a_{ij} + \sum_{k=1}^{\infty}\theta_{ij}^{(k)}\mu^k]y_j, & i = 1, 2, \cdots \infty, \\ y_i \quad = \sum_{j=1}^{\infty} A_j e^{-at}[x_{ij}^{(0)}(t) + \sum_{k=1}^{\infty} x_{ij}^{(k)}(t)\mu^k], & i = 1, 2, \cdots \infty. \end{cases}$$

On imposing the conditions that the y_i be periodic with the period 2π, viz., $y_i(2\pi) - y_i(0) = 0$, we get

$$(43) \qquad 0 = \sum_{j=1}^{\infty} A_j[e^{-2a\pi}x_{ij}^{(0)}(2\pi) - c_{ij} + e^{-2a\pi}\sum_{k=1}^{\infty}x_{ij}^{(k)}(2\pi)\mu^k], \quad i = 1, 2, \cdots \infty.$$

After setting $e^{-2a\pi} = \sigma/\sigma - 1$, the equations (43) become

$$(44) \qquad 0 = \sum_{j=1}^{\infty} A_j[\sigma\{x_{ij}^{(0)}(2\pi + \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi)\mu^k\} - c_{ij}(\sigma - 1)] = 0, \quad i = 1, 2, \cdots \infty.$$

To avoid the trivial case where the A_j are all zero, we must set the determinant

$$(45) \qquad D(\sigma, \mu) = |\ [\sigma\{x_{ij}^{(0)}(2\pi) + \sum_{k=1}^{\infty}x_{ij}^{(k)}(\pi)\mu^k\} - c_{ij}(\sigma - 1)]\ | = 0,$$

which is a condition on the undetermined constant a. Since the $x_{ij}^{(0)} + \sum_{k=1}^{\infty} x_{ij}^{(k)}\mu^k$ are the elements of a fundamental set, and $c_{ii} = 1$, it follows that $D(\sigma, \mu)$ converges absolutely for all finite σ's. If the fundamental equation is satisfied when $a = a_0$, it is also satisfied when $a = a_0 + \nu\sqrt{-1}$, where ν is any integer; but all distinct solutions of the differential equations can be obtained by taking $\nu = 0$, since the ratios of the A_j are the same for $\nu = 0$ as for $\nu = p$. And for $\mu = 0$ the equation (45) reduces to

$$(46) \qquad D_0 = |\ \sigma x_{ij}^{(0)}(2\pi) - c_{ij}(\sigma - 1)\ | = 0.$$

When $\mu = 0$, we have the case of constant coefficients which we treated in II. Now we shall determine the c_{ij} and the $a^{(0)} = \lambda^{-1}$ of the solutions as we did there. But having the c_{ij} we might also determine the a of the solutions by means of (46). Since for given initial conditions the solution of the differential equations is unique, and since the initial conditions are the same in the two cases, the solutions obtained by the former method and the solution obtained by means of (46) are the same.

We recall that the c_{ij} are not all zero if and only if the characteristic equation has finite roots. Therefore for every root of the characteristic equation there is a corresponding root σ of (46); and conversely, for every root of (46) there is as corresponding root of the characteristic equation. There-

are zero for $\sigma = \sigma_1$. Hence we can solve equations (37) for $B_2, B_3, \cdots,$ terms of $y_{i_1}(0)$. Consequently in this case we get a second solution associated with a_1, which is of the form (34).

In a similar manner we can go ahead step by step and get the following group of solutions associated with a_1

$$\begin{cases} x_{i_1} = e^{a_1 t} y_{i_1}, \\ x_{i_2} = e^{a_1 t}[y_{i_2} + t y_{i_1}], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{i_n} = e^{a_1 t}[y_{i_n} + t y_{i_{n-1}} + \cdots + 1/(n-1)!\, t^{n-1} y_{i_1}]. \end{cases}$$

If σ_1 is a triple root of $D(\sigma) = 0$, such that all first minors are zero, but not all the second minors are zero, the solutions associated with a_1 are

$$x_{i_1} = e^{a_1 t} y_{i_1},\ x_{i_2} = e^{a_1 t} y_{i_2},\ x_{i_3} = e^{a_1 t}[y_{i_3} + t(y_{i_1} + y_{i_2})].$$

All the sub-cases can be treated, as they arise, by the methods given here.

IV

Now let us assume that the system of differential equations (1) satisfy the hypotheses of the second existence theorem, and in addition the coefficients in the power series expansions of the θ_{ij} are separately periodic with the period 2π. That theorem tells us that the solutions of (1) can be written in the form

$$(39) \qquad x_{ij} = \sum_{k=1}^{\infty} x_{ij}^{(k)}(t)\mu^k, \quad i, j = 1, 2, \cdots \infty.$$

And we shall take the initial conditions such that

$$(40) \qquad x_{ij}(0) = \sum_{k=1}^{\infty} x_{ij}^{(k)}(0)\mu^k \equiv c_{ij},$$

whence

$$x_{ij}^{(0)}(0) = c_{ij},\ x_{ij}^{(k)}(0) = 0,\ k = 1, 2, \cdots \infty.$$

where the c_{ij} are constants such that $c_{ii} = 1$, and their determinant is absolutely convergent. These conditions coupled with the fact that if the determinant of a set of solutions converges and is not zero when $t = 0$, it converges and is not zero for every value of t for which $\sum_{i=1}^{\infty} \theta_{ii}$ converges, show that the system (40) constitutes a fundamental set of solutions.

Now we inquire if we can find solutions of (1) of the form

$$(41) \qquad x_i = e^{at} y_i,$$

where the y_i are periodic with the period 2π, and a is a constant which remains

$$\begin{vmatrix} \sigma\sum_{j=1}^{\infty} A_j\phi_{1j} - (\sigma - 1), & \sigma\phi_{12} & \cdots \\ \sigma\sum_{j=1}^{\infty} A_j\phi_{2j} - A_2(\sigma - 1), & 1 + \sigma(\phi_{22} - 1), & \cdots \\ \cdots\cdots, & \cdots\cdots, & \cdots \end{vmatrix} = D(\sigma) = 0.$$

So we have

$$(32)\quad D(\sigma) = (1 - \sigma/\sigma_1)\, D_1 = (1 - \sigma/\sigma_1) \begin{vmatrix} y_{11}(0), & \sigma\phi_{12}, & \cdots \\ y_{21}(0), & 1 + \sigma(\phi_{22} - 1), & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

Since $\sigma = \sigma_1$ is an n-fold root of $D(\sigma) = 0$, $D_1(\sigma)$ has the factor $(1 - \sigma/\sigma_1)^{n-1}$. Since (27) constitutes a fundamental set, any solution can be expressed in the form

$$(33)\qquad x_i = B_1 e^{a_1 t} y_{i_1} + \sum_{j=2}^{\infty} B_j \phi_{ij}, \quad i = 1, 2, \cdots, \infty.$$

Now let us make the transformation, corresponding to (19), to get a second solution associated with a_1.

$$(34)\qquad x_{i_2} = e^{a_1 t}(y_{i_2} + t y_{i_1}), \quad i = 1, 2, \cdots \infty.$$

On imposing the condition that x_{i_2} shall satisfy the system (1), we find since $e^{a_1 t} y_{i_1}$ is a solution,

$$(35)\qquad y'_{i_2} + a_1 y_{i_2} = \sum_{j=2}^{\infty} \theta_{ij}(t) y_{j_2} - y_{i_1}, \quad i = 1, 2, \cdots \infty.$$

From the form of (35) we see that sufficient conditions that the y_{i_2} shall be periodic with the period 2π are

$$(36)\qquad y_{i_2}(2\pi) - y_{i_2}(0) = 0 = \sum_{j=2}^{\infty} B_j[e^{-2a_1\pi}\phi_{ij}(2\pi) - \phi_{ij}(0)] \\ - 2\pi y_{i_1}(0) = 0.$$

On substituting $e^{-2a_1\pi} = \sigma_1/\sigma_1 - 1$, (36) becomes

$$(37)\qquad -2\pi(\sigma_1 - 1) y_{i_1}(0) + \sum_{j=2}^{\infty}{}' B_j \phi_{ij} + B_k[1 + \sigma_1(\phi_{kk} - 1] = 0, \quad k \neq j.$$

The condition that these equations be consistent is, since $\sigma_1 \neq 1$,

$$D_1(\sigma) = \begin{vmatrix} y_{11}(0), & \sigma_1\phi_{12}, & \cdots \\ y_{21}(0), & 1 + \sigma_1(\phi_{22} - 1), & \cdots \\ \cdots\cdots, & \cdots\cdots, & \cdots \end{vmatrix} = 0.$$

In (32) we showed that D_1 vanishes $(n - 1)$ times when $\sigma = \sigma_1$. By hypotheses not all the first minors corresponding to the elements of the first column

infinite number of finite roots, the system (1) has a fundamental set of solutions, each of the elements of which is of the form (19).

The discussion of the form of the solutions follows precisely the same lines as that in the case of constant coefficients except when σ_0 is an n-fold root of (26) and not all the $(n-k)$th, $k>0$, minors of $D(\sigma)$ vanish for $\sigma=\sigma_0$.

Now let us assume that $\sigma=\sigma_0$ is an n-fold root of (26) and not all the first minors of $D(\sigma)$ vanish. Then there is only one solution of (1) of the form $x_i=e^{\alpha_1 t}y_i$, where the y_i are expressed as in (21). And let us choose the notation so that a minor corresponding to the elements of the first column is not zero.

Then we take as a new set of solutions

$$(27)\qquad x_{i1}=e^{\alpha_1 t}y_{i1},\ x_{ij}=\phi_{ij}(t),\ i=1,2,\cdots\infty,\ j=2,3,\cdots\infty.$$

This set of solutions can be shown to constitute a fundamental set in a precisely similar fashion to that in II. Then we make the transformation

$$x_i=e^{at}z_i,\ i=1,2,\cdots\infty.$$

As above we get

$$(28)\qquad z_i=e^{-at}[A_1e^{\alpha_1 t}y_{i1}+\sum_{j=2}^{\infty}A_j\phi_{ij}(t)],\ i=1,2,\cdots\infty.$$

Necessary and sufficient conditions that the z_i be periodic with the period 2π are

$$z_i(2\pi)-z_i(0)=0,\ i=1,2,\cdots\infty.$$

On imposing these conditions on (28). we get

$$(29)\qquad A_1[e^{-2(a-a_1)\pi}y_{i1}(2\pi)-y_{i1}(0)]+\sum_{j=2}^{\infty}A_j[\phi_{ij}(2\pi)e^{-2a\pi}-\phi_{ij}(0)]=0.$$

After setting $e^{-2a\pi}=\sigma/\sigma-1$, the equations (29) become

$$(30)\qquad A_1(1-\sigma/\sigma_1)y_{i1}(0)+\sum_{j=2}^{\infty}{}'A_j\sigma\phi_{ij}+A_k[1+\sigma(\phi_{kk}-1)]=0,$$
$$k\neq j,\ k=2,3,\cdots\infty,\ i=1,2,\cdots\infty.$$

The fundamental equation for the equations (30) is

$$(31)\qquad \begin{vmatrix}(1-\sigma/\sigma_1)y_{11}(0), & \sigma\phi_{12}, & \cdots\\ (1-\sigma/\sigma_1)y_{21}(0), & 1+\sigma(\phi_{22}-1), & \cdots\\ \cdot & \cdot & \cdot\end{vmatrix}=0.$$

Making use of (21) and taking $A_1=1$, equation (31) becomes

Since the system ϕ_{ij} constitutes a fundamental set of solutions of (1), any solution of (20) can be written

$$(21) \qquad y_i = e^{-at}\sum_{j=1}^{\infty} A_j\phi_{ij}(t), \quad i = 1, 2, \cdots \infty.$$

We now inquire whether it is possible to determine the A_j and a so that the y_i defined by (21) shall be periodic with the period 2π. From the form of (20) it is evident that necessary and sufficient conditions that the y_i be periodic with the period 2π are.

$$(22) \qquad y_i(2\pi) - y_i(0) = 0, \quad i = 1, 2, \cdots \infty.$$

On imposing these conditions on (21), we get

$$(23) \qquad \sum_{j=1}^{\infty} A_j[e^{-2a\pi}\phi_{ij}(2\pi) - \phi_{ij}(0)] = 0, \quad i = 1, 2, \cdots \infty.$$

Then we make the transformation

$$(24) \qquad e^{-2a\pi} = \sigma/\sigma - 1,$$

assuming of course that $e^{-2a\pi} \neq 1$. Then the equations (23) become

$$(25) \qquad \sum_{j=1}^{\infty}{}' A_j\sigma\phi_{ij}(2\pi) + A_i[1 + \sigma(\phi_{ii}(2\pi) - 1] = 0, \qquad i \neq j, \; i = 1, 2, \cdots \infty.$$

In order that these equations have a solution other than that in which the A_j are all zero, it is necessary that the determinant of the coefficients be zero. On writing the $\phi_{ij}(2\pi)$ simply ϕ_{ij}, the determinant is

$$(26) \qquad D(\sigma) = \begin{vmatrix} 1 + \sigma(\phi_{11} - 1), & \sigma\phi_{12}, & \cdots \\ \sigma\phi_{21}, & 1 + \sigma(\phi_{22} - 1), & \cdots \\ \cdots\cdots, & \cdots\cdots, & \cdots \end{vmatrix} = 0.$$

This determinant, we see from the fact that the determinant of ϕ_{ij} converges absolutely, is an integral function of σ. Equation (26) is called the *fundamental equation* associated with the period 2π, and does not admit $\sigma = 0$ as a root.

As in the characteristic equation in II, the fundamental equation may have no finite root, a finite number of finite roots, or a denumerably infinite number of finite roots.

As in the case of constant coefficients, if the fundamental equation has no roots, the system (1) has no solution of the form (19), if the fundamental equation has a finite number of finite roots, the system (1) has a finite number of solutions of the form (19), and finally if the fundamental equation has an

Then by a precisely similar argument to the foregoing we can show that

$$x_{i1}, x_{i2}, \phi_{ij}, \quad i=1, 2, \cdots \infty, \quad j=3, 4, \cdots \infty,$$

constitute a fundamental set.

When for a given value of a the characteristic equation has a root of higher multiplicity, discussions similar to the foregoing must be made.

We should expect to treat next the case that $\Delta(\lambda)=0$ has a root of infinite multiplicity. But such a case cannot arise. For we have seen that $\Delta(\lambda)$ is an integral transcendental function of λ. Therefore it can be expanded in a Taylor's series in the neighborhood of any finite λ. Suppose that λ_0 were a finite root of infinite multiplicity. Then $\Delta(\lambda)$ vanishes together with all of its derivatives when $\lambda=\lambda_0$. Hence $\Delta(\lambda)$ vanishes at every point in the neighborhood of λ_0. Then it is identically zero. But this is not true, for when $\lambda=0$, $\Delta(\lambda)=1$. Hence $\Delta(\lambda)$ has no root of infinite multiplicity.

If $a=0$, we can not make the transformation which carries the system (12) into the system (13), and the determinant of (12) diverges. Then we do not know whether the equations (12) have a solution for the c_j not all zero; but each special case must be considered as it arises.

To sum up our results we have

Theorem IV. *If and only if $\Delta(\lambda)$ has an infinite number of roots λ_i, the system (1) has a fundamental set of solutions each of whose elements is of the form*

$$x_{ij}=e^{a_j t}\psi_{ij}(t), \quad i, j=1, 2, \cdots \infty,$$

where $a_1=-1/\lambda_i$ and the ψ_{ij} are polynomials in t of degree at most $(n-1)$, and n is the order of multiplicity of the root λ_j.

III

Now we shall assume that the θ_{ij} of Theorem I are periodic with the period 2π. We have seen that the system (1) has, as a fundamental set of solutions, the set $\phi_{ij}(t)$, where

$$\phi_{ii}(0)=1, \quad \phi_{ij}(0)=0, \quad i\neq j, \quad i, j, =1, 2, \cdots \infty. \tag{18}$$

Let us make the transformation

$$x_i=e^{at}y_i, \tag{19}$$

where a is an undetermined constant. Then the equations (1) become

$$y_i'+ay_i=\sum_{j=1}^{\infty}\theta_{ij}(t)y_j, \quad i=1, 2, \cdots \infty. \tag{20}$$

Next we make the transformation $y_2 = d_1 y_1 + d_2 x_2 + \cdots$, $y'_2 = d_1 y'_1 + d_2 x'_2 + \cdots$ and determine, if possible, $d_2, d_3, \cdots$, such that $y'_2 = a y_2 + y_1$. Then, as in the foregoing, we get the following, infinite set of equations for the determination of the d's

$$(17') \begin{cases} (a_1 - a) d_1 + a_{21} d_2 + a_{31} d_3 + \cdots = 1, \\ 0 + (a_{22} - a_{21} m_2 - a) d_2 + \cdots = 0, \\ \cdots\cdots\cdots\cdots \end{cases}$$

On setting $a = -1/\lambda$, we get an equivalent set of equations for the determination of the d's, the determinant of which is

$$\Delta_1(\lambda) = \begin{vmatrix} 1 + \lambda a_1, \lambda a_{21}, & \\ 0, 1 + \lambda (a_{22} - m_2 a_{21}), \cdots \\ \cdots\cdots\cdots \end{vmatrix} = (1 + \lambda a_1) \begin{vmatrix} 1 + \lambda (a_{22} - a_{21} m_2), \cdots \\ \lambda (a_{32} - a_{21} m_3), \cdots \\ \cdots\cdots\cdots \end{vmatrix}$$

By making use of the equations (17), the reader will convince himself that $\Delta_1(\lambda)$ is identically equal to $\Delta(\lambda)$, for it is obtained from $\Delta(\lambda)$ first by interchanging the rows and columns of $\Delta(\lambda)$ and then multiplying the elements of certain rows by certain quantities and adding them to the corresponding elements of other rows. Furthermore there is a first minor of $\Delta_1(\lambda)$ which is not zero when $a = a_1$. This minor involves the elements of the first row, since all the other first minors vanish.

The determinant of the coefficients of the equations, omitting the first one, is zero for $a = a_1$, since when $a = a_1$, $\Delta(\lambda)$ has a multiple root λ_1. Therefore we can solve for the ratios of $d_2, d_3, \cdots$. Then let us substitute the values of the d's thus obtained in the first equation of (17'). First, we know that the sum will converge *; for the d's are proportional to the first minors of $\Delta_1(\lambda)$, and the sum of the products of the elements of a row and the corresponding first minors converges. Secondly, since the d's which we have determined carry an arbitrary factor, we can finally determine them so that the first equation is satisfied.

Therefore our equations have been reduced to

$$(1'') \quad y'_1 = a_1 y_1, \; y'_2 = y_1 + a_1 y_2, \; x'_3 = a^{(2)}_{31} y_1 + a^{(2)}_{32} y_2 + a^{(2)}_{33} x_3 + \cdots, \cdots.$$

where the $a^{(2)}_{ij}$ are the transformed a_{ij} of the original equations. And on solving the systems (1') and (1'') and putting in place of y_1 and y_2 the corresponding value of x_1 and x_2, we see that the solutions of (1) associated with a_1 are

$$x_{i1} = c_{i1} e^{a_1 t}, \; x_{i2} = (c_{i2} + t c_{i1}) e^{a_1 t}, \; i = 1, 2, \cdots \infty.$$

* F. Riesz, "Les systèmes d'équations linéaires à une infinité d'inconnues," p. 34.

that $\binom{1\ 2}{1\ 2}$ is not zero, we get two independent solutions of the form

$$x_{i1} = c_{i1}e^{a_1 t},\ x_{i2} = c_{i2}e^{a_1 t},\ i = 1, 2, \cdots \infty,$$

which with ϕ_{ij}, $j = 3, 4, \cdots \infty$, constitute a fundamental set; for

$$\begin{vmatrix} x_{11}, & x_{12}, & \phi_{13}, & \cdots \\ x_{21}, & x_{22}, & \phi_{23}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots \end{vmatrix} = \begin{vmatrix} A_1^{(1)}\phi_{11} + A_2^{(1)}\phi_{12}, & A_1^{(2)}\phi_{11} + A_2^{(2)}\phi_{12}, & \phi_{13}, & \cdots \\ A_1^{(1)}\phi_{21} + A_2^{(1)}\phi_{22}, & A_1^{(2)}\phi_{21} + A_2^{(2)}\phi_{22}, & \phi_{23}, & \cdots \\ \cdots\cdots\cdots\cdots, & \cdots\cdots\cdots\cdots, & \cdots, & \cdots \end{vmatrix}.$$

and the $A_2^{(1)}$, $A_1^{(2)}$ may be taken to be zero, and $A_1^{(1)} = A_2^{(2)} = 1$. So these solutions, we see, constitute a fundamental set.

Next let us assume that not all the first minors of $\Delta(\lambda)$ vanish when $a = a_1 = a_i$; and let us choose the notation so that the first minor $\binom{1}{1}$ is not zero. Then to get a solution associated with a_1, we make the transformation $y = b_1x_1 + b_2x_2 + \cdots$, $y' = b_1x'_1 + b_2x'_2 + \cdots$ and, if possible, determine the b's in such a manner that $y' = ay$. On substituting these equations in (1) we get $b_1[a_{11}x_1 + a_{12}x_2 + \cdots] + b_2[a_{21}x_1 + a_{22}x_2 + \cdots] + \cdots = a[b_1x_1 + b_2x_1 + \cdots]$.
This equation must hold for all initial values of the x's. Therefore it is an identity in them; and the b's satisfy the following set of equations

$$(17)\qquad \begin{cases} (a_{11} - a)b_1 + a_{21}b_2 + \cdots = 0, \\ a_{12}b_1 + (a_{22} - a)b_2 + \cdots = 0, \\ \cdots\cdots\cdots\cdots\cdots \end{cases}$$

In order that these equations have a solution for the b's not all zero, we get as a necessary and sufficient condition, just as in the case of c's in (13), $\Delta(\lambda) = 0$. This condition is satisfied, for we have assumed that when $a = a_1$, the characteristic equation is satisfied. Furthermore we have chosen the notation such that the minor $\binom{1}{1}$ is not zero when $a = a_1$. Then we get for the b's

$$b_j = m_j b_1,\quad j = 1, 2, \cdots \infty,$$

where $m_j = \dfrac{\binom{1}{j}}{\binom{1}{1}}$. For convenience we shall take $b_1 = 1$. Then the equations become

$$(1')\begin{cases} y'_1 = a_1 y_1, \\ x'_2 = a_{21}y_1 + (a_{22} - a_{21}m_2)x_2 + (a_{23} - a_{21}m_3)x_3 + \cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ x'_n = a_{n1}y_1 + (a_{n2} - a_{n1}m_2)x_2 + (a_{n3} - a_{n1}m_3)x_3 + \cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Theorem III: *If the θ_{ij} of* (1) *are constants and if λ_0 is a root of $\Delta(\lambda)=0$ such that all minors of order $r-1$, but not all of order r, vanish for $\lambda=\lambda_0$, then there exist r independent solutions of* (1) *of the form* (11).

Now the characteristic equation, since it is an integral transcendental function, may have no finite roots, it may have a finite number of finite roots, or a denumerably infinite number of roots. In the first case the c_i are all zero, in the second case, a finite number of solutions of the form (11) exists and in the last case an infinite number of the form (11) exists. This is the case we shall study, for this infinite set constitutes a fundamental set of solutions as the following discussion shows.

For each λ that is a root of $\Delta(\lambda)=0$ there is a solution of (1) of the form

$$(16) \qquad x_{ij}=c_{ij}e^{a_j t}, \quad i, j=1, 2, \cdots \infty.$$

We know that the system (1) admits the fundamental set of solutions ϕ_{ij}, $i, j=1, 2, \cdots \infty$, where $\phi_{ii}(0)=1$, $\phi_{ij}(0)=0$, $i\neq j$. Consequently

$$x_{i1}=\sum_{k=1}^{\infty} A_k^{(1)}\phi_{ik}, \quad i=1, 2, \cdots \infty.$$

First let us assume that the a_j are all distinct. Then it follows immediately that the system (16) constitutes a fundamental set. For consider x_{i1}, ϕ_{ij}, $i=1, 2, \cdots \infty$, $j=2, 3, \cdots \infty$.

$$\begin{vmatrix} \sum_{k=1}^{\infty} A_k^{(1)}\phi_{1k}, & \phi_{12}, & \cdots \\ \sum_{k=1}^{\infty} A_k^{(1)}\phi_{2k}, & \phi_{22}, & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots \end{vmatrix} = A_1^{(1)} \begin{vmatrix} \phi_{11}, & \phi_{12}, & \cdots \\ \phi_{21}, & \phi_{22}, & \cdots \\ \cdots, & \cdots, & \cdots \end{vmatrix}$$

Choose the initial conditions such that $A_1^{(1)}=1$, which is no actual restriction since we are merely determining the arbitrary constant which enters, and we can multiply our solution by a constant, not zero, when we are through if we choose. Thus x_{i1}, ϕ_{ij}, $i=1, 2, \cdots \infty$, $j=2, 3, \cdots \infty$, constitute a fundamental set. Similarly we show that x_{i1}, x_{i2}, ϕ_{ij}, constitute a fundamental set, where $j=3, 4, \cdots \infty$. Continuing in this manner and passing to the limit we see that the x_{ij}, $i, j=1, 2, \cdots \infty$, constitute a fundamental set. A result of this determination of the arbitrary constant is that $c_{ii}=1$.

Next let $a_2=a_1$, and $a_j\neq a_1$, $j=3, 4, \cdots \infty$. There are two cases that may arise here, viz., all of the first minors of $\Delta(\lambda)$ vanish when $a=a_1$, or not all vanish. Consider the former case. Then (15) assures us that we can solve for the c_{ij} in terms of two of them. On choosing the notation so

ditions that such a solution exist are

$$(12) \qquad \begin{cases} (a_{11} - a)c_1 + a_{12}c_2 + a_{13}c_3 + \cdots = 0, \\ a_{21}c_1 + (a_{22} - a)c_2 + a_{23}c_3 + \cdots = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

In order to see that (12) can have a solution for which the c_i are not all zero, we make the transformation $a = -\lambda^{-1}$, assuming of course that a is not zero. Then the equations (12) become

$$(13) \qquad \begin{cases} (1 + \lambda a_{11})c_1 + \lambda a_{12}c_2 + \cdots = 0, \\ \lambda a_{21}c_1 + (1 + \lambda a_{22})c_2 + \cdots = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

In order that the equations (13) have a solution in which the c_i are not all zero, it is necessary and sufficient that λ be a root of

$$(12) \qquad \Delta(\lambda) = \begin{vmatrix} 1 + \lambda a_{11}, & \lambda a_{12}, & \cdots \\ \lambda a_{21}, & 1 + \lambda a_{22}, & \cdots \\ \cdots\cdots, & \cdots\cdots, & \cdots \end{vmatrix} = 0,$$

Equation (14) is called the *characteristic equation*. Since the a_{ij} satisfy the hypotheses of Theorem I, it follows that $\Delta(\lambda)$ is an integral function * of λ.

If λ_0 is a root of (14), then there exists a finite number r such that $\Delta(\lambda_0)$ together with its minors of order $1, 2, \cdots, r-1$, vanishes; but there is at least one minor of order r which is different from zero.† Let the minor which is obtained by replacing the elements in the i_1th row and the k_1th column by unity and the remaining elements in that row and column by zero, and so on up to the i_rth row and the k_rth column be denoted by

$$\begin{pmatrix} i_1, & i_2, & \cdots, & i_r \\ k_1, & k_2, & \cdots, & k_r \end{pmatrix}^{(\lambda_0)},$$

and suppose that this minor is one which is different from zero. Then the system (13) admits the solution ‡

$$(15) \quad c_k = \frac{\begin{pmatrix} i_1, & \cdots & i_r \\ k_1, & \cdots & k_r \end{pmatrix}^{(\lambda_0)} \xi_{k_1} + \begin{pmatrix} i_1, & i_2 & \cdots & i_r \\ k_1, & k & \cdots & k_r \end{pmatrix}^{(\lambda_0)} \xi_{k_2} + \cdots + \begin{pmatrix} i_1, & \cdots & i_r \\ k_1, & \cdots & k \end{pmatrix}^{(\lambda_0)} \xi_{k_r}}{\begin{pmatrix} i_1, & \cdots & i_r \\ k_1, & \cdots & k_r \end{pmatrix}^{(\lambda_0)}}$$

depending on r independent parameters $\xi_{k_1}, \cdots \xi_{k_r}$, and admits no others.

To sum up our results we have the following

---

* See K, p. 104. † See K, p. 108. ‡ See K, p. 109.

We see that the determinant of the ϕ_{ij} is absolutely convergent.

Theorem II. *Suppose that the system* (1) *satisfies the following hypotheses:*

(H_1) *The θ_{ij} are expansible as power series in a parameter μ, which converge for all real $|t| < R$ if $|\mu| < \rho$.*

(H_2) *For $\mu = 0$, the $\theta_{ij} = a_{ij}$, where the a_{ij} are constants.*

(H_3) *The θ_{ij} satisfy all the hypotheses of Theorem I uniformly with respect to μ if $|\mu| < \rho$.*

Then the solutions of (1) *can be expressed as power series in μ, which converge for $|\mu| \leqq \rho_0 < \rho$ and for all real $|t| < R$.*

In Theorem I we showed that $x_i - \beta_i = \sum\limits_{j=1}^{\infty} \beta_i^{(j)} t^j$, $|t| < R$, $|\mu| < \rho$, where $n\beta_i^{(n)} = P_i^{(n)}(\beta_j^{(0)}, \cdots, \beta_j^{(n-1)})$.

In particular $\beta_i^{(1)} = \sum\limits_{j=1}^{\infty} a_{ij}^{(0)} \beta_j^{(0)}$. The $a_{ij}^{(0)}$ are power series in μ. Hence $\beta_j^{(0)} a_{ij}^{(0)} = \sum\limits_{k=1}^{\infty} \beta_{ijk}^{(0)} \mu^k$, $\beta_i^{(1)} = \sum\limits_{j=1}^{\infty}\sum\limits_{k=1}^{\infty} \beta_{ijk}^{(0)} \mu^k$, which converge for $|\mu| < \rho$. Then by a well-known theorem in the theory of double series, we infer that $\beta_i^{(1)} = \sum\limits_{k=1}^{\infty} \beta_{ik}^{(1)} \mu^k$, $|\mu| < \rho$. In a similar fashion we proceed step by step, and show that the $\beta_i^{(n)}$ are also power series in μ, which converge for $|\mu| < \rho$. So we have

$$x_i = \sum_{k=1}^{\infty} \phi_{ik}(\mu), \quad i = 1, 2, \cdots \infty, \tag{10}$$

which is uniformly convergent for $|\mu| \leqq \rho_0 < \rho$, and where $\phi_{ik}(\mu) = \phi_{ik}^{(0)} + \phi_{ik}^{(1)}\mu + \cdots$. Then by a well-known theorem of Weierstrass, we know that we can rearrange the series (10), and write it in the form

$$x_i = \sum_{j=1}^{\infty} x_i^{(j)}(t)\mu^j, \quad |\mu| \leqq \rho_0 < \rho, \; |t| < R, \; i = 1, 2, \cdots \infty.$$

II

Now let us assume that the θ_{ij} of Theorem I are constants; and let these constants be denoted by a_{ij}. We inquire whether the system (1) has a solution of the form

$$x_i = c_i e^{at}, \tag{11}$$

where by (H_3) of Theorem I $\sum\limits_{j=1}^{\infty} a_{ij} c_j$ converges. Necessary and sufficient con-

To prove the convergence of (2), consider the system of differential equations

$$(5)\qquad \xi_i' = S_i \sum_{j=1}^{\infty} T_j \xi_j, \quad i = 1, 2, \cdots \infty.$$

Upon substituting $\xi_i = \beta_i + \sum_{j=1}^{\infty} \gamma_i^{(j)} t^j$ in (5) and equating coefficients, we get

$$(6)\qquad \begin{cases} \gamma_i^{(1)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(0)}, \\ 2\gamma_i^{(2)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(1)}, \\ \cdots\cdots\cdots \\ (\lambda+1)\gamma_i^{(\lambda+1)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(\lambda)}, \text{ where } \gamma_i^{(0)} = \beta_i. \end{cases}$$

On comparison of (6) and (4) we see that $\gamma_i^{(j)} > |\beta_i^{(j)}|$, $i, j = 1, 2, \cdots \infty$. Hence ξ_i dominates x_i for every i.

From (5) we have

$$(7)\qquad \frac{1}{S_1}\xi_1' = \frac{1}{S_2}\xi_2' = \cdots = \xi'.$$

On taking $\xi(0) = 0$, we have

$$(8)\qquad \xi_i - \beta_i = S_i \xi, \quad i = 1, 2, \cdots \infty.$$

Therefore each equation (5) reduces to

$$(9)\qquad \xi' = C\xi + K,$$

where $C = S_1T_1 + S_2T_2 + \cdots$, $K = \beta_1 T_1 + \beta_2 T_2 + \cdots$.
From the theory of a finite system of differential equations we know that (9) has a unique analytic solution for $|t| < R$. On combination of this fact with (8), we see that the solution (2) of (1) converges when $|t| < R$.

Now we define as a fundamental set of solutions of (1), a set such that every solution of (1) can be expressed as linear homogeneous functions with constant coefficients of the elements of the set. Then if we denote by ϕ_{ij} the elements of the fundamental set, it follows that the determinant of the ϕ_{ij} converges and is not zero for all $|t| < R$, and conversely. Furthermore it has the form $\Delta = \Delta_0 e^{\int \sum_{i=1}^{\infty} \theta_{ii}(t)\,dt}$, for all $|t| < R$, where Δ_0 is the value of the determinant Δ when $t = t_0$. If, for example, we take the solutions defined by $\phi_{ii}(0) = 1$ and $\phi_{ij}(0) = 0$, $i \neq j$, we see from (8) and (9) that

$$|\phi_{ii}| \leqq 1 + \frac{S_i T_i}{C} |(e^{Ct} - 1)|, \text{and } |\phi_{ij}| \leqq \frac{S_i T_j}{C} |(e^{Ct} - 1)|, i \neq j.$$

I

The differential equations which we shall consider are of the general type

$$(1) \qquad x_i' = \sum_{j=1}^{\infty} \theta_{ij}(t) x_j, \quad i = 1, 2, \cdots \infty,$$

where x_i' is the derivative of x_i with respect to the independent variable t.

Concerning this system of differential equations we shall establish two existence theorems.

Theorem I. *Suppose the system* (1) *satisfies the following hypotheses:*
(H_1) *The* $\theta_{ij}(t)$ *are analytic functions of* t *when* $|t| < R$.
(H_2) *Positive constants,* $S_1, S_2, \cdots, T_1, T_2, \cdots$, *exist such that* $|\theta_{ij}(t)| < S_i T_j, |t| < R$, and $S_1 T_1 + S_2 T_2 + \cdots$ *converges.*
(H_3) *The* $x_i(0) = \beta_i$, *where the* β_i *are constants such that* $\beta_1 T_1 + \beta_2 T_2 \cdots$ *converges.*

Then there exists a unique system of functions

$$(2) \qquad x_i = \beta_i + \sum_{j=1}^{\infty} \beta_i^{(j)} t^j, \quad i = 1, 2, \cdots \infty,$$

which satisfy the system (1), *and which converge for* $|t| < R$.

Although this theorem was established by von Koch,* it will be proved briefly here because the notation and the results are essential for the later parts of this paper.

From (H_1) we have

$$(3) \qquad \theta_{ij}(t) = a_{ij}^{(0)} + a_{ij}^{(1)} t + \cdots, \quad i, j = 1, 2, \cdots \infty.$$

Upon substituting the series (2) in (1) and equating coefficients, we get

$$(4) \qquad \begin{cases} \beta_i^{(1)} = \sum_{j=1}^{\infty} a_{ij}^{(0)} \beta_j^{(0)}, \\ 2\beta_i^{(2)} = \sum_{j=1}^{\infty} (a_{ij}^{(1)} \beta_j^{(0)} + a_{ij}^{(0)} \beta_j^{(1)}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (\lambda + 1) \beta_i^{(\lambda+1)} = \sum_{j=1}^{\infty} \sum_{k=0}^{\lambda} a_{ij}^{(\lambda-k)} \beta_j^{(k)}, \text{ where } \beta_i^{(0)} = \beta_i. \end{cases}$$

We see that the formal solution is unique.

---

* *Loc. cit.*

On the Solution of Certain Types of Linear Differential Equations in Infinitely Many Variables.

By Webster G. Simon.

The main purpose of this paper is to prove the existence of certain types of solutions of particular kinds of linear differential equations with periodic coefficients in infinitely many variables. As a means to this end the existence of exponential solutions is established for certain types of linear differential equations in infinitely many variables with constant coefficients.

The starting point is the existence theorem given by von Koch,* and a generalization of Poincaré's theorem † concerning the development of the solutions of the differential equations as power series in a parameter μ when the functions appearing in the differential equations are themselves power series in μ. Then our work is very similar to that of the finite case, the finite determinants becoming infinite determinants which together with all their first minors converge absolutely.

The type of determinant used in this paper is more general than the normal determinant, and is the following.§ The determinant

$$\begin{vmatrix} 1+a_{11}, & a_{12}, & a_{13}, & \cdots \\ a_{21}, & 1+a_{22}, & a_{23}, & \cdots \\ a_{31}, & a_{32}, & 1+a_{33}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots \end{vmatrix}$$

is such that there exist two sets of positive constants $S_1, S_2, \cdots, T_1, T_2, \cdots$, which are of such a nature that $|a_{ij}| < S_i T_j$, and $\sum_{i=1}^{\infty} S_i T_i$ converges. This is a type of infinite determinant given by von Koch,§ which together with all its minors converges absolutely.

Using the methods thus indicated, we find that many of the phenomena of the finite systems are carried over into the infinite systems of differential equations.||

* Von Koch, Ofversigt af Kongliga Vetenskaps Akademiens Förhandlingar, Vol. 56 (1899), pp. 395-411.

† Poincaré, "Les Méthodes Nouvelles de la Mécanique Céleste," Vol. I, chapter II.

§ Von Koch, *Acta Mathematica*, Vol. 24 (1901), pp. 89-122. (Hereafter this paper will be denoted by K.

|| See Moulton and MacMillan, American Journal of Mathematics, Vol. 33 (1911), pp. 63-96.

the Green's function for a square with the point of discontinuity at the center and diagonal of length 2:

$$\begin{aligned} G &= -\log r - .28 - .14\, r^4 \cos 4\theta - .01\, r^8 \cos 8\theta + .002\, r^{12} \cos 12\theta + \cdots \\ &= Re\,(-\log z - .28 - .14\, z^4 - .01\, z^8 + .002\, z^{12} + \cdots\cdots) \qquad (25) \end{aligned}$$

approximately.

Thus we see that the expansion for u, that is, $G + \log r$ is the real part of an expansion in fourth powers of z. This may be verified by examining the expressions in elliptic functions, obtained by the method of images, for the square with the point of discontinuity and the origin of co-ordinates at the center. The character of the expansion for the Green's function for the square suggests immediately that for a regular polygon of p sides (p any positive integer) with discontinuity at the center, u will be the real part of an expansion in pth powers. It is evident at once that this is true when p becomes infinite, for in that case we have a circle for which u is constant. This may be shown to be true in general by making use of Schwarz' mapping function for a regular polygon of p sides.

From the second of equations (22) we have $C_{2n}=0$ for every n. Thus we obtain a new set of functions Φ', all of which are orthogonal to Φ_0, as follows:

$$\begin{array}{ll}
\Phi'_1 = R\cos\theta & \Phi'_2 = R\sin\theta \\
\Phi'_3 = R^2\cos 2\theta & \Phi'_4 = R^2\sin 2\theta \\
\Phi'_5 = R^3\cos 3\theta & \Phi'_6 = R^3\sin 3\theta \\
\Phi'_7 = R^4\cos 4\theta - .48826 & \Phi'_8 = R^4\sin 4\theta \\
\Phi'_9 = R^5\cos 5\theta & \Phi'_{10} = R^5\sin 5\theta \\
\cdots & \cdots \\
\Phi'_{15} = R^8\cos 8\theta - 1.11259 & \Phi'_{16} = R^8\sin 8\theta \\
\cdots & \cdots \\
\Phi'_{23} = R^{12}\cos 12\theta - 3.01549 & \Phi'_{24} = R^{12}\sin 12\theta \\
\cdots & \cdots
\end{array}$$

Now we take Φ'_1 and orthogonalize Φ'_2, Φ'_3, Φ'_4, $\cdots\cdots$ to it in the same way as before. This process is continued until a sufficient number of orthogonal functions has been obtained. Finally these orthogonal functions must be normalized by dividing each function Φ by $\left[\int_0^{2\pi}\Phi^2\,d\theta\right]^{\frac{1}{2}}$. Denote by Ψ_0, Ψ_1, Ψ_2, $\cdots\cdots$, as in the preceding section, the normal, orthogonal functions obtained in this way.

Next we determine the A's by means of the formula

$$A_m = \int_0^{2\pi}\log R(\theta)\Psi_m(\theta)\,d\theta.$$

Finally, substituting in equations (18), we have the required constants for the square, namely,

$$\begin{aligned}
c_0 &= \sum_{i=0}^{\infty} a_{i0}A_i = -.23653 - .05516 + .00883 + .00474 + \cdots = -.28\cdots \\
c_1 &= c_2 = c_3 = c_4 = c_5 = c_6 = 0 \\
c_7 &= \sum_{i=7}^{\infty} a_{i7}A_i = -.11296 - .02100 - .00834 + \cdots = -.14\cdots \\
c_8 &= c_9 = c_{10} = c_{11} = c_{12} = c_{13} = c_{14} = 0 \\
c_{15} &= \sum_{i=15}^{\infty} a_{i15}A_i = -.00794 - .00013 + \cdots = -.01\cdots \\
c_{16} &= c_{17} = c_{18} = c_{19} = c_{20} = c_{21} = c_{22} = 0 \\
c_{23} &= \sum_{i=23}^{\infty} a_{i23}A_i = +.00153 + \cdots = +.002\cdots \\
&\cdots
\end{aligned}$$

Putting these constants in equation (5), we have the following formula for

$$\psi_2 = a_{20} + a_{21} r \cos\theta + a_{22} r \sin\theta$$
$$\psi_3 = a_{30} + a_{31} r \cos\theta + a_{32} r \sin\theta + a_{33} r^2 \cos 2\theta$$
$$\cdots\cdots\cdots\cdots\cdots$$

Now substituting for the coefficients A_i from the equation (15), we have

$$G = -\log r + \sum_{i=0}^{\infty} \psi_i(r,\theta) \int_0^{2\pi} \log R(\theta)\, \Psi_i(\theta)\, d\theta, \tag{21}$$

which, since the matrix (a_{ik}) is limited, can be written

$$G = -\log r + \int_0^{2\pi} \log R(t) \sum_{i=0}^{\infty} [\psi_i(r,\theta)\, \Psi_i(t)]\, dt. \tag{22}$$

Thus we have an expression for the Green's function in terms of a *line integral around the boundary.* When $r = R(\theta)$, $\psi_i(r,\theta) = \Psi_i(\theta)$, and the last member of the equation (21) becomes the expansion for $\log R$ in terms of the Ψ's, so that G vanishes identically.

Application to a Square Contour.

As a simple application of the above, let us find the Green's function for a square with discontinuity at the center. If we make the apothem of the square unity and measure angles from it, we have for the function R

$$\begin{aligned}
R(\theta) &= \sec\theta & -\pi/4 &= &&= \pi/4 \\
R(\theta) &= \csc\theta & \pi/4 &= &&= 3\pi/4 \\
R(\theta) &= -\sec\theta & 3\pi/4 &= &&= 5\pi/4 \\
R(\theta) &= -\csc\theta & 5\pi/4 &= &&= 7\pi/4.
\end{aligned} \tag{23}$$

Beginning with the first function $\Phi_0 = 1$ we shall first orthogonalize all the other functions to it by determining C_{2n-1} and C_{2n} to satisfy the following equations:

$$\begin{aligned}
\int_0^{2\pi} (\Phi_{2n-1} - C_{2n-1}\Phi_0)\Phi_0\, d\theta &= 0 \\
\int_0^{2\pi} (\Phi_{2n} - C_{2n}\Phi_0)\Phi_0\, d\theta &= 0.
\end{aligned} \tag{24}$$

From the first of these equations we have $C_{2n-1} = 0$ for n odd and $n/2$ odd; that is,

$$C_1 = C_3 = C_5 = C_9 = C_{11} = C_{13} = C_{17} \cdots\cdots\cdots = 0.$$

For $n/2$ even

$$C_{2n-1} = 2/\pi \int_{\pi/4}^{\pi/4} \sec^n\theta \cos n\theta\, d\theta.$$

$$\int_0^{2\pi} R^{2n} \cos n\theta \, d\theta = b^2_{2n-1,\,0} + b^2_{2n-1,\,1} + \cdots\cdots + b^2_{2n-1,\,2n-1} \tag{19}$$
$$\int_0^{2\pi} R^{2n} \sin n\theta \, d\theta = b^2_{2n,\,0} + b^2_{2n,\,1} + \cdots\cdots + b^2_{2n,\,2n}.$$

Hence for every k, we have

$$b^2_{2n-1,\,k} < \int_0^{2\pi} R^{2n} \cos n\theta \, d\theta < \int_0^{2\pi} R^{2n} \, d\theta = 2\pi R^{2n}_{max},$$
$$b^2_{2n,\,k} < \int_0^{2\pi} R^{2n} \sin n\theta \, d\theta < \int_0^{2\pi} R^{2n} \, d\theta = 2\pi R^{2n}_{max};$$

so that we have, for the norm of the sequence $\{b_{ik}\}$,

$$\sum_{i=k}^{\infty} b^2_{ik} < 4\pi \sum_{2n=k}^{\infty} R^{2n}_{max}. \qquad k = 0,\ 1,\ 2,\ \cdots\cdots \tag{20}$$

But the maximum value of $R(\theta)$ is always less than unity. Hence the right hand member of equation (20) is an absolutely convergent series. Thus we have proved that $\{b_{ik}\}$ is of finite norm for every k.

(2) The second condition is that the matrix (a_{ki}) of the coefficients in equations (18) shall be limited. This we can show to be true as follows: Consider first the matrix (b_{ik}) of equations (13). From (19) we have

$$\sum_{i=0}^{\infty} \sum_{k=0}^{i} b^2_{ik} < 4\pi \sum_{2n=k}^{\infty} R^{2n}_{max} < M, \text{ a constant};$$

that is, the b's are of finite norm and hence* the matrix (b_{ik}) is limited. Now the matrix (a_{ik}) of equations (12) is the unique reciprocal of the matrix (b_{ik}), and hence is itself limited.† Therefore its conjugate, the matrix (a_{ki}) of equations (17) is limited.

(3) The third condition is evidently satisfied; for the sequence $\{A_i\}$ of the right hand members of equation (17) is of finite norm since the A's are the Fourier constants for the expansion of $\log R(\theta)$ in terms of the Ψ's.

Now by Mrs. Pell's theorem it follows that equations (18) give an actual solution of equation (17), and hence give us the required expressions for the constants c.

Formula (5) for the Green's function may now be written

$$G = -\log r + A_0\psi_0 + A_1\psi_1 + A_2\psi_2 + A_3\psi_3 + \cdots\cdots\cdots,$$

where

$$\psi_0 = a_{00}$$
$$\psi_1 = a_{10} + a_{11} r \cos\theta$$

---

* Hellinger and Toeplitz: Math. Ann., Vol. 69, p. 307.

† Hellinger and Toeplitz: Math. Ann., *l. c.*, p. 311.

for each c in the form of an infinite series involving the b's and the A's. If, then, we substitute for the b's from equation (14), we arrive at the simple result

$$\begin{aligned}
c_0 &= a_{00}A_0 + a_{10}A_1 + a_{20}A_2 + a_{30}A_3 + a_{40}A_4 + \cdots\cdots\cdots \\
c_1 &= \qquad\quad\; a_{11}A_1 + a_{21}A_2 + a_{31}A_3 + a_{41}A_4 + \cdots\cdots\cdots \\
c_2 &= \qquad\qquad\qquad\quad a_{22}A_2 + a_{32}A_3 + a_{42}A_4 + \cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (18)\\
c_m &= \sum_{i=m}^{\infty} a_{im}A_i \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Substituting this solution in equation (6), we have

$$\sum_{i=0}^{\infty} a_{i0}A_i + \sum_{n=1}^{\infty}\left(\sum_{i=2n-1}^{\infty} a_{i,\,2n-1}A_iR^n\cos n\theta + \sum_{i=2n}^{\infty} a_{i,\,2n}A_iR^n\sin n\theta\right) = \sum_{i=0}^{\infty} A_i\Psi_i.$$

It is sufficient to multiply equations (12) by $A_0, A_1, A_2, \cdots\cdots$ respectively, and add, to verify formally the correctness of this equation. Now since we know the c's exist and are unique, if we can prove that equations (18) give an actual solution of (17), this is the only solution. To prove this we shall make use of the following theorem due to Anna J. Pell: *

Theorem: If the sequences $\{\lambda_k\}$ and $\{\mu_i\}$ are such that the sequence $\{b_{ik}/\lambda_k\}$ is of finite norm for every k, and the matrix $(\lambda_i a_{ki}/\mu_i)$ is limited, then for every sequence $\{A_k\}$ such that $\{\mu_k A_k\}$ is of finite norm, the system of equations

$$\sum_{i=k}^{\infty} b_{ik}c_i = A_k \qquad (k = 0, 1, 2, 3, \cdots\cdots)$$

has a solution c_k such that $\{\lambda_k c_k\}$ is of finite norm, and the solution is given by

$$c_k = \sum_{i=k}^{\infty} a_{ik}A_i \qquad (k = 0, 1, 2, 3, \cdots\cdots)$$

Making $\lambda_k = \mu_i = 1$, we can show that the three conditions required by the above theorem are satisfied, as follows:

(1) The first condition to be satisfied is that the sequence $\{b_{ik}\}$ shall be of finite norm for every k; that is, the coefficients in equations (13), taken by columns, must give sequences of finite norm. This we can prove as follows: Since the Ψ's are normal, orthogonal functions we have, for every n,

* Ann. of Math., Vol. 28, p. 35. We have changed Mrs. Pell's notation to conform to our own.

814737

Solving for the Φ's, we have

$$
\begin{aligned}
\Phi_0 &= \quad 1 \quad = b_{00}\Psi_0 \\
\Phi_1 &= R\cos\theta = b_{10}\Psi_0 + b_{11}\Psi_1 \\
\Phi_2 &= R\sin\theta = b_{20}\Psi_0 + b_{21}\Psi_1 + b_{22}\Psi_2 \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\Phi_{2n-1} &= R^n\cos n\theta = b_{2n-1,\,0}\Psi_0 + b_{2n-1,\,1}\Psi_1 + b_{2n-1,\,2} + \cdots\cdots \\
&\qquad\qquad + b_{2n-1,\,2n-1}\Psi_{2n-1} \\
\Phi_{2n} &= R^n\cos n\theta = b_{2n,\,0}\Psi_0 + b_{2n,\,1}\Psi_1 + b_{2n,\,2} + \cdots\cdots \\
&\qquad\qquad + b_{2n,\,2n-1}\Psi_{2n-1} + b_{2n,\,2n}\Psi_{2n} \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad ,
\end{aligned}
\tag{13}
$$

where

$$
{}_m = (-1)^{n+m}\frac{1}{a_{m,\,m}\,a_{m+1,\,m+1}\cdots\cdots a_{nn}}
\begin{vmatrix}
a_{m+1,\,m} & a_{m+1,\,m+1} & 0 & 0 & \cdots\cdots & 0 \\
a_{m+2,\,m} & a_{m+2,\,m+1} & a_{m+2,\,m+2} & 0 & \cdots\cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & 0 \\
a_{n-1,\,m} & a_{n-1,\,m+1} & a_{n-1,\,m+2} & a_{n-1,m+3} & \cdots\cdots & a_{n-1,\,n-1} \\
a_{n,\,m} & a_{n,\,m+1} & a_{n,\,m+2} & a_{n,m+3} & \cdots\cdots & a_{n,\,n-1}
\end{vmatrix}
\quad (
$$

Now expand $\log R(\theta)$ in terms of $\Psi_0, \Psi_1, \Psi_2, \cdots\cdots$ thus:

$$
\log R(\theta) = \sum_{i=0}^{\infty} A_i\Psi_i,
$$

where the coefficients are given by

$$
A_i = \int_0^{2\pi} \log R(\theta)\, \Psi_i(\theta)\, d\theta. \tag{15}
$$

Substituting in equation (6), we have

$$
c_0\,(b_{00}\Psi_0) + c_1\,(b_{10}\Psi_0 + b_{11}\Psi_1) + c_2\,(b_{20}\Psi_0 + b_{21}\Psi_1 + b_{22}\Psi_2) + \cdots\cdots = \sum_{i=0}^{\infty} A_i\Psi_i, \tag{16}
$$

where $c_0, c_1, c_2, \cdots\cdots$ are the unknown constants required. Comparing coefficients of the Ψ's on each side of the equation, we have

$$
\begin{aligned}
b_{00}\,c_0 + b_{10}\,c_1 + b_{20}\,c_2 + b_{30}\,c_3 + \cdots\cdots\cdots\cdots &= A_0 \\
b_{11}\,c_1 + b_{21}\,c_2 + b_{31}\,c_3 + \cdots\cdots\cdots\cdots &= A_1 \\
b_{22}\,c_2 + b_{32}\,c_3 + \cdots\cdots\cdots\cdots &= A_2 \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \quad ,
\end{aligned}
\tag{17}
$$

an infinite set of equations for the c's. Solving this set of equations formally for each c by eliminating the other c's one at a time, we have an expression

functions which will be orthogonal for the variable θ between 0 and 2π. These functions we can construct by the method of Goursat, provided the functions are linearly independent. We shall first investigate this matter of linear independence.

An infinite set of functions $\Phi_n(\theta)$ are said to be linearly independent if no finite number of them are linearly dependent; that is, if no equation of the form

$$C_0\Phi_{r_0} + C_1\Phi_{r_1} + C_2\Phi_{r_2} + C_3\Phi_{r_3} + \cdots\cdots + C_n\Phi_{r_n} = 0, \tag{11}$$

where the c's are constants not all zero, exists between a finite number of them. For the set of functions we are considering there certainly do exist such relations for some contours $r = R(\theta)$. For example, if $R = \sec\theta$ we have

$$\Phi_0 - \Phi_1 = 0; \qquad 2\Phi_2 - \Phi_4 = 0.$$

However, for no closed finite contour can such a relation exist. For, suppose we consider a contour $r = R(\theta)$ such that a linear relation of the form (11) exists. Then the left hand member of this equation is the value $V(R, \theta)$ of a harmonic function $V(r, \theta)$ on a finite closed contour; hence by Gauss's mean value theorem,

$$V(r, \theta) \equiv 0$$

for all points inside the contour, which is only possible if all the coefficients vanish. Hence the set of functions

$$\Phi_0 = 1,\ \Phi_1 = R\cos\theta,\ \Phi_2 = R\sin\theta,\ \Phi_3 = R^2\cos 2\theta,\ \cdots\cdots,$$

and in fact any infinite set $\Phi_n(\theta)$ obtained in the same way from a linearly independent set of two-dimensional harmonics $\phi_n(r, \theta)$, are linearly independent for *all closed contours*, in particular for polygons.

We can now apply Goursat's method and form a set, $\Psi_0, \Psi_1 \ldots\ldots$, of normal, orthogonal functions, as follows:

$$\begin{aligned}
\Psi_0 &= a_{00}\Phi_0 \\
\Psi_1 &= a_{10}\Phi_0 + a_{11}\Phi_1 \\
\Psi_2 &= a_{20}\Phi_0 + a_{21}\Phi_1 + a_{22}\Phi_2 \\
\Psi_3 &= a_{30}\Phi_0 + a_{31}\Phi_1 + a_{32}\Phi_2 + a_{33}\Phi_3 \\
&\cdots\cdots\cdots\cdots \\
\Psi_m &= a_{m0}\Phi_0 + a_{m1}\Phi_1 + a_{m2}\Phi_2 + a_{m3}\Phi_3 + \cdots\cdots + a_{mm}\Phi_m \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{12}$$

$\sin n\chi$, the function $H(\chi)$ is nothing else than the value of u on the unit circle,

$$H(\chi) = u(1, \chi);$$

and the kernel reduces to the well-known expansion for

$$\frac{1 - R^2}{1 - 2R\cos(\chi - \theta) + R^2}. \qquad R < 1.$$

Thus the integral equation (9) now becomes

$$1/2 \int_0^{2\pi} u(1, \chi) \frac{1 - R^2(\theta)}{1 - 2R(\theta)\cos(\chi - \theta) + R^2(\theta)} d\chi = \log R(\theta), \tag{10}$$

where the left hand member is Poisson's integral. Hence we may state the solution of the problem before us as follows:

Consider a closed contour of such a nature that a circle, with center at a point 0, can be drawn completely enclosing it, but not enclosing any of the images of 0 with respect to the contour or arcs composing the contour. Make this the unit circle. Then the Green's function for this contour is obtained by using Poisson's integral backwards, as it were, to determine the value on the unit circle, namely $H(\chi) = u(1, \chi)$, of a harmonic function $u(r, \theta)$ which reduces to $\log R(\theta)$ for points of the given contour. Having found H we determine the Fourier constants or coefficients corresponding to it and substitute them in formula (5) to get the Green's function.

On account of the complicated character of the kernel in equation (10) the solution of this equation seems to be impracticable. We shall therefore take up in the following pages an entirely different method for determining the constants for equation (5.)

Practical Solution by an Infinite Set of Equations.

Going back to equation (6), it is possible to reduce the problem of determining the unknown constants c_m to the solution of an infinite number of equations for this infinite set of unknowns. One way to do this would be to expand the various powers of $R(\theta)$ as well as the function $\log R(\theta)$ in Fourier series and compare the coefficients of $\cos n\theta$ and $\sin n\theta$ on each side of the equation. This leads immediately to an infinite set of equations for the constants c_m, but these equations are very awkward to handle.

In order to get a more easily manageable set of equations we will construct a normalized, orthogonal set of functions from the set

$$\Phi_0 = 1,\ \Phi_1 = R(\theta)\cos\theta,\ \Phi_2 = R(\theta)\sin\theta,\ \Phi_3 = R^2(\theta)\cos 2\theta, \cdots\cdots,$$

for the Green's function for any convex polygon whatever, with the point of discontinuity at any point whatever inside the polygon. Evidently this method can be applied, not only to polygons, but to *any convex contour whatever* and the point of discontinuity may be taken at *any point whatever* inside the contour.

The whole problem is then reduced to the method of determining the infinite set of constants A_n and B_n, or c_m. This problem may be solved in two ways: (1) by integral equations and (2) by an infinite set of algebraic equations. The first method gives only a formal solution; the second is a practical method as we shall show.

Formal Solution by Integral Equations.

The c's are Fourier constants. Hence, by a well-known theorem of Fischer and Riesz, the sum of their squares $\sum_0^\infty c^2_m$, converges. Moreover, if we select any set of orthogonal functions whatever, linear, planar or solid, for example $h_m(\chi)$, there exists one and only one function $H(\chi)$ for which c_m are the Fourier constants with respect to this orthogonal set. Whether the series thus obtained converges or actually represents the function at all is immaterial; for in any case the formulas for the coefficients hold, namely:

$$c_m = \int_{\chi_1}^{\chi_2} H(\chi)\ h_m(\chi)\ d\chi, \tag{8}$$

where the definite integral is either single, double or triple, according as the orthogonal functions chosen are linear, planar or solid. Now substitute these constants back in equation (6) and we have, provided the series in the bracket converges,

$$\int_{\lambda_1}^{\lambda_2} H(\chi)\ \left[\sum_0^\infty h_m(\chi)\,{}_m\Phi_m(\theta)\ \right] d\chi = \log R(\theta), \tag{9}$$

an integral equation of the first kind with unsymmetric kernel $K(\chi,\theta) = \sum_0^\infty h_m(\chi)\Phi_m(\theta)$. This integral equation has been studied by Picard, Schmidt, Bateman and others, and existence theorems have been worked out; but no general method of solution has yet appeared in the mathematical literature. The following special form of the equation, however, is of considerable interest:

If we take for the set of orthogonal functions $h_m(\chi)$ the linear set $\cos n\chi$,

where z is a complex variable. Since u is harmonic the expression $\sum_1^\infty (A_n - iB_n) z^n$ is analytic in a circle extending to the nearest image $0'$ of 0. Consider for example the polygon* $ABCDE$ (Fig. 3) with the point of discontinuity at a point 0 inside it. By means of Borel's integral we can continue the function $\sum_0^\infty A_n z^n$ analytically to the region enclosed by the polygon

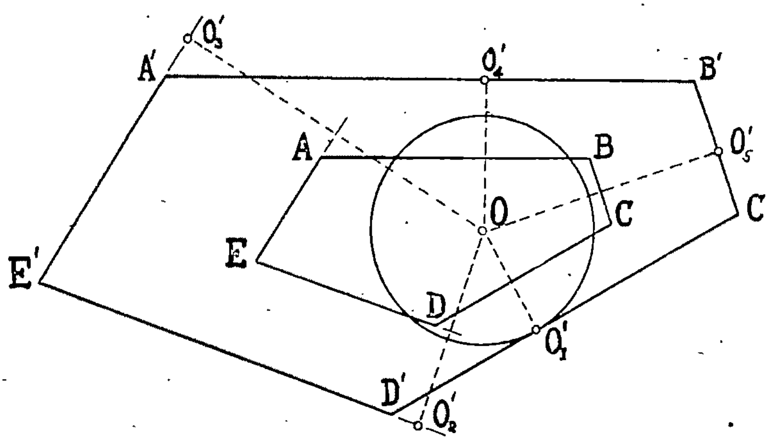


FIG. 3.

$A'B'C'D'E'$ whose sides are lines through the several images $0'$ and perpendicular respectively to the several lines $00'$. The expression for the function thus extended is

$$f(z) = \int_0^\infty e^{-t} \sum_0^\infty \frac{A_n (zt)^n}{n!} dt.$$

Applying this to the problem in hand, we have, in the real plane,

$$u = A_0/2 + Re \int_0^\infty e^{-t} \sum_1^\infty \frac{(A_n - iB_n)\ (zt)^n}{n!} dt$$

$$= A_0/2 + \int_0^\infty e^{-t} \sum_1^\infty (A_n r^n \cos n\theta + B_n r^n \sin n\theta) \frac{t^n}{n!} dt.$$

This function is harmonic inside the polygon $A'B'C'D'E'$ whose sides are respectively parallel to the sides of $ABCDE$. Hence this gives us a formula,

$$G = -\log r + A_0/2 + \int_0^\infty e^{-t} \sum_1^\infty (A_n r^n \cos n\theta + B_n r^n \sin n\theta) \frac{t^n}{n!} dt, \quad (7)$$

* Some or all of the sides of the polygon could be replaced by analytic arcs.

expansion in the whole circle? To answer this question we cite the following theorem due to Bôcher.*

Theorem: If u is a function harmonic throughout the neighborhood of the point x_0y_0 and if, when this function is continued analytically, the distance from x_0y_0 to the nearest singular point of u, which lies in the same sheet of the Riemann surface generated by the analytic continuation in which x_0y_0 lies, is K, then the development † in polar co-ordinates (the point x_0y_0 being taken as pole);

$$u = A_0/2 + \Sigma\,(A_n r^n \cos n\theta + B_n r^n \sin n\theta),$$

converges and represents u throughout the interior of the circle of radius $r = K$ and does not converge throughout any continuum which does not lie in this circle.

If we restrict the problem, then, to contours of such a nature that a point 0 can be found whose images all lie outside of a circle about 0 enclosing the contour (this we can make the unit circle), we have from the above theorem, for all contours of this type and all such points 0 within each contour, the Green's function for the point 0 and the given contour in the form

$$\begin{aligned} G &= -\log r + A_0/2 + \sum_1^\infty (A_n r^n \cos n\theta + B_n r^n \sin n\theta) \\ &= -\log r + \sum_0^\infty c_m \phi_m(r, \theta). \end{aligned} \tag{5}$$

This expansion holds for all points inside the polygon and on the boundary. Moreover it is unique and we must be able to determine the constants A_n and B_n or c_m from the boundary conditions.

For the boundary we have $r = R(\theta)$ and $G = 0$, that is,

$$A_0/2 + \sum_1^\infty [A_n R^n(\theta) \cos n\theta + B_n R^n(\theta) \sin n\theta] = \log R(\theta)$$

or

$$\sum_0^\infty c_m \phi_m(\Phi) = \log R(\theta), \tag{6}$$

where $\Phi_0 = 1$; $\Phi_{2n-1} = R^n \cos n\theta$; $\Phi_{2n} = R^n \sin n\theta$.

Now we can apply a method of analytic continuation due to Borel, to obtain a new expression for the function u which will hold for a much more extended region, as follows: Equation (3) may be written

$$u = A_0/2 + Re \sum_1^\infty (A_n - iB_n)\, z^n,$$

---

* Trans. Am. Math. Soc., Vol. 10, p. 278.

† Bôcher gives here also an equivalent in terms of polynomials in x and y.

well-known theorem we can expand the harmonic function u uniquely in the series,

$$u = A_0/2 + \sum_{1}^{\infty} r^n \,(A_n \cos n\theta + B_n \sin n\theta), \tag{3}$$

Where A_n and B_n are the Fourier constants for the area enclosed by the circle: $0 \leqq \theta \leqq 2\pi$, $0 \leqq r \leqq r_0$. The functions $r^n \cos n\theta$ and $r^n \sin n\theta$ form a complete set of orthogonal functions for the circular boundary. Let us write equation (3) for convenience in the form

$$u = \sum_{m=0}^{\infty} c_m \phi_m(r, \theta), \tag{4}$$

where $\phi_0 = 1$, $\phi_1 = r\cos\theta$, $\phi_2 = r\sin\theta$, ...

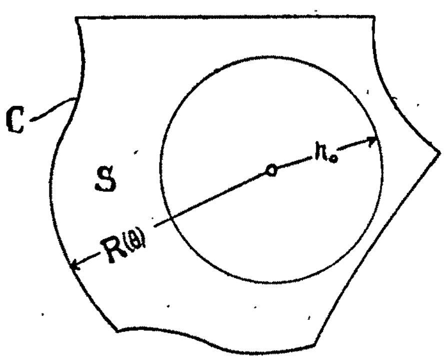


FIG. 1.

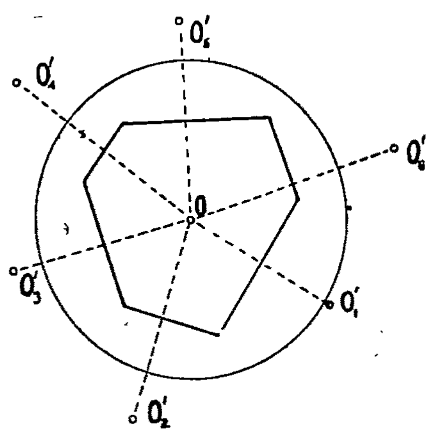


FIG. 2.

We can continue the function u analytically across the boundary C (Fig. 1) by means of Schwarz' principle of symmetry. The extended function thus obtained takes values equal but opposite in sign at every pair of points which are symmetrical * with respect to an arc or segment of the contour. At the symmetrical points or images $0'$ of the point 0, the function becomes infinite as log r, but at all other symmetrical points it is harmonic. Hence if we construct the images † of the point 0 in the several arcs or segments of the contour, we know that the function u exists and is harmonic at least in a circle extending to the nearest image. Fig. 2 is drawn for a special case where the contour is a polygon.

Now, the question arises: Is the function u represented by the same

* For definition see Osgood, p. 671.

† For the method see Lery's thesis on *La Fonction de Green pour une Contour Algébrique*, p. 49.

set of equations was simple. In the problem before us the solution is much more involved, though entirely manageable in a large number of cases.

This method of procedure is much more widely applicable than either of the two methods described above. These two methods give workable results for, at most, the regular polygons, two other polygons, the five-pointed star, and the rectangle. Our method is applicable, not only to all regular polygons, but to a large class of irregular polygons as well, and to many other contours. It gives the Green's function in the form, log $1/r$ plus an expansion in circular harmonics, $r^n \cos n\theta$ and $r^n \sin n\theta$. The coefficients of this expansion are each given in the form of an infinite series of constants and can therefore be determined to any desired degree of accuracy by taking a sufficient number of terms. We are convinced that the method is actually workable in a practical sense in a large number of cases.

We have confined our attention to a single class of Green's functions, but the same procedure might be employed to get the Green's function for any differential equation with any given boundary conditions.* It might also be extended to problems in three dimensions.

Formulation of the Problem.

By a plane contour we mean any closed plane curve which is regular in the sense of Osgood; that is, is composed of a finite number of analytic arcs or straight segments. An important case is that of the polygon.

The Green's function for any such contour is given by the following formula in polar co-ordinates:

$$G(r, \theta) = -\log r + u(r, \theta), \tag{1}$$

where the function u is harmonic, that is, satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

and has no singularities in the region.

Consider this function, $u(r, \theta)$. We know that it is everywhere harmonic in the area S enclosed by the contour C and has no singularities in S. Moreover, on the boundary, for which $r = R(\theta)$ and $u = U(\theta)$, we have $G = 0$ and

$$U(R, \theta) = \log R(\theta). \tag{2}$$

Draw any circle, radius r_0 (Fig. 1), about 0 so that it lies wholly in S. By a

* See Hadamard's Columbia lectures, pp. 47-52.

any given polygon can be mapped conformally on to the unit circle. Knowing this mapping function $f(z)$, we can obtain the Green's function for the polygon immediately from the formula:

$$G = -Re \log f(z) \qquad (Re \equiv \text{real part of})$$

By this method we are able to get the Green's function for the equilateral triangle, the 45 degree right-triangle, an infinite strip, a regular five-pointed star, and a rectangle. Beyond this it is difficult to go. Other polygons lead to Abelian integrals and hyper-elliptic functions. Moreover, except for regular polygons, it seems to be impossible to determine the unknown constants which appear under the integral sign in the function $f(z)$. For regular polygons, however, we could evaluate Schwarz' integral,

$$w = \int_0^z \frac{dz}{(1-z^n)^{\frac{2}{n}}},$$

in the form of an infinite series, and take the real part of the logarithm. In other words, the series obtained by expanding

$$G = -Re \log \int_0^z \frac{dz}{(1-z^n)^{\frac{2}{n}}},$$

gives us the value of the Green's function for a regular polygon of n sides in a circle of radius

$$r_0 = \int_0^1 \frac{dz}{(1-z^n)^{\frac{2}{n}}},$$

where the point of discontinuity is at the center of the polygon. This work of Schwarz has been considerably discussed and amplified by other writers, particularly in a recent book by Study.* However, very little of fundamental importance has been added to Schwarz' original memoirs.

The method of arriving at the Green's function which we shall employ in this article is new, although in principle similar to the methods used by Fourier and Neuman in attacking various physical problems. We select an infinite set of linearly independent functions which are solutions of the given differential equation, and seek to expand the Green's function, or rather, the quantity $G-\gamma$, where γ is the principal solution ($\log 1/r$ in this case), in a series of these functions, determining the coefficients of the series so that the boundary conditions are satisfied. The determination of these coefficients depends upon the solution of an infinite set of equations in an infinity of unknowns. In the problems solved by Fourier the solution of the infinite

* *Vorlesungen über Ausgewählte Gegenstände der Geometrie*, 2tes Heft.

The Green's Function for a Plane Contour.

By Hobart Dickinson Frary.

A Green's function is a function of a boundary and two points. The points, of which one is considered fixed and the other variable, both lie inside (or both outside) the boundary. In two dimensions the boundary is a contour. A large part of mathematical physics is devoted to the solution of boundary problems in two dimensions. All such problems can be solved immediately if the Green's function for the given boundary or contour is known.

The character of the Green's function depends on three things,—(1) the differential equation of which it is a solution, (2) the contour or boundary to which it applies, and (3) the boundary conditions. The class of Green's functions to which we shall confine ourselves in this article is the one of most frequent occurrence in mathematical physics, namely, the Green's function for Laplace's differential equation, with what Hilbert calls boundary condition (I); that is, $G=0$ on the boundary. Having found this particular solution G, we can solve the problem of Dirichet in the plane; that is, we can determine a solution u satisfying any given condition of the form $u=U(\theta)$ on the boundary. This we can do by making use of the formula

$$u=1/2\pi \int_{c} U \frac{\partial G}{\partial n} ds,$$

which in polar co-ordinates becomes

$$u(r,\theta)=1/2\pi \int_{0}^{2\pi} U(\theta) \left[-r\frac{\partial G}{\partial r}d\theta+\frac{\partial G}{r\partial\theta}dr\right]$$

There are two well recognized methods of obtaining the Green's function in two dimensions, both of which, however, are very limited in their applications. These methods are the following:

(1) *The method of images.* This method is easily applicable to the circle, semi-circle, infinite strip, half plane, equilateral triangle, 30 degree right-triangle, and the rectangle; but for other contours it would involve complicated Riemann surfaces and hyper-elliptic functions. Apparently no one has attempted to apply the method to other polygons.

(2) *The method of Schwarz.* Schwarz obtained a general formula $w=f(z)$, where $f(z)$ is in the form of a definite integral, by means of which

sary substitution is found to be

$$p+2z=\frac{\sqrt{mn}\,(1-\kappa w)}{1+\kappa w},$$

where

$$m=p+2z_0,\ n=p+2c-2z_0,\ \kappa=\frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}.$$

Since p, c, and z_0 are all positive, then m is also positive. If $z_0<p/2+c$ then n is likewise positive, and therefore κ is real and less than 1 numerically. When the above transformation is made in (10′) we obtain

$$-\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}\int\frac{(1-\kappa w)\,dw}{(1-\kappa w)\sqrt{(1-w^2)(1-\kappa^2w^2)}}=2\sqrt{g}(t-t_0),$$

which simplifies to

$$\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}\left\{I_1-2I_3-\frac{2\kappa}{1-\kappa^2}\sqrt{\frac{1-w^2}{1-\kappa^2w^2}}\right\}=2\sqrt{g}(t-t_0),$$

where

$$I_1=\int\frac{dw}{\sqrt{(1-w^2)(1-\kappa^2w^2)}},$$

an elliptic integral of the first kind, and

$$I_3=\int\frac{dw}{(1-\kappa^2w^2)\sqrt{(1-w^2)(1-\kappa^2w^2)}},$$

an elliptic integral of the third kind. Thus t can be found for any given value of w or z. For most purposes, however, it is more convenient to have z expressed as a function of t. This will be discussed in the next section.

The half-period, $T/2$, of a complete oscillation can be obtained by integrating (10′) between the limits z_0 and $c-z_0$. The corresponding limits for w are -1 and $+1$. Hence

$$\begin{aligned}2\sqrt{g}(T/2)=\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}&\left\{\int_{-1}^{+1}\frac{dw}{\sqrt{(1-w^2)(1-\kappa^2w^2)}}\right.\\&-2\int_{-1}^{+1}\frac{dw}{(1-\kappa^2w^2)\sqrt{(1-w^2)(1-\kappa^2w^2)}}\\&\left.-\frac{2\kappa}{1-\kappa^2}\left[\sqrt{\frac{1-w^2}{1-\kappa^2w^2}}\right]_{-1}^{+1}\right\}.\end{aligned}$$

Since $\left[\sqrt{\dfrac{1-w^2}{1-\kappa^2 w^2}}\right]_{-1}^{+1}=0,$ then

$$T=\frac{4\sqrt{mn}}{\sqrt{g}(\sqrt{m}+\sqrt{n})}\left\{\int_0^1\frac{dw}{\sqrt{(1-w^2)(1-\kappa^2w^2)}}\right.$$
$$\left.-2\int_0^1\frac{dw}{(1-\kappa^2w^2)\sqrt{(1-w^2)(1-\kappa^2w^2)}}\right\}.$$

§ 5. *Construction of a Periodic Solution for the Vertical Motion.* Let

$$z=v+c/2, \tag{12}$$

where v is the new dependent variable. Then the last equation of (8) becomes

$$v''=-\frac{4gv+v'^2}{k+2v},$$

where $k=p+c$. This equation can be expanded in the form

$$v''=-(1/k)(4gv+v'^2)[1-2v/k+4v^2/k^2$$
$$+\cdots(-2v/k)^j+\cdots], \tag{13}$$

which will converge for $|v|<|k/2|$, or $z<p/2+c$. Hence the particle must not rise above the plane $z=p/2+c$, being the plane at the distance c above the horizontal plane passing through the focus of the generating parabola of (7).

Now let

$$v=\gamma w,\ t-t_0=\tfrac{1}{2}\sqrt{k/g}\ (1+\delta)\ \tau, \tag{14}$$

where γ and δ are arbitrary parameters. When (14) is substituted in (13), the factor γ can be cancelled off in the resulting equation and we obtain

$$\ddot{w}=-[(1+\delta)w+\mu\dot{w}^2][1-2\mu w+\cdots(-2\mu w)^j+\cdots], \tag{15}$$

where $\mu=\gamma/k$, and the dots denote derivation with respect to τ.

The terms of (15) which are independent of μ and δ are $\ddot{w}+w=0$, and the period of the solution is 2π in τ. Now an existence proof, based on Poincaré's extension to Cauchy's theorem, would show that a periodic solution of (15) exists having the form,

$$w=\sum_{j=0}^{\infty}w_j\mu^j,\quad \delta=\sum_{j=1}^{\infty}\delta_j\mu^j, \tag{16}$$

where the w_j are periodic functions of τ with the period 2π, and the δ_j are constants; and that this solution converges for all values of τ in the interval $0\leqq\tau\leqq2\pi$, provided $|\mu|$ is sufficiently small. Instead of making this rather

simple existence proof, we shall assume the form of solution (16), make the formal construction, and then show that Macmillan's theorem will apply to prove the convergence of the solution.

Since

$$z = v + c/2 = \gamma w + c/2 = k\mu w + c/2, \tag{17}$$

and since $\dot{z}' = 0$ at $t = t_0$, then at $\tau = 0$ (or $t = t_0$) we may choose,

$$w(0) = a,\ \dot{w}(0) = 0,$$

where a is a real constant. When these initial conditions are imposed on (16), we have

$$w_0(0) = a,\ w_j(0) = 0,\ j = 1, \cdots \infty,\ \dot{w}_j(0) = 0,\ j = 0, 1, \cdots \infty. \tag{18}$$

Let (16) be substituted in (15) and let the resulting equation be denoted by (15′). This equation is an identity in μ and we may equate the coefficients of the same powers of μ, thus obtaining differential equations which define the various w_j. These equations are to be integrated, and the constants of integration and the various δ_j are to be chosen so that the solutions will be periodic and satisfy the initial conditions (18).

Step 0: *Coefficient of* μ^0. The terms of (15′) that are independent of μ give the equation $\ddot{w}_0 + w_0 = 0$, and its solution which satisfies (18) is $w_0 = a\cos\tau$.

Step 1: *Coefficient of* μ. The differential equation obtained from the terms in μ is

$$\ddot{w}_1 + w_1 = -\delta_1 w_0 + 2w_0^2 - \dot{w}_0^2 = -a\delta_1\cos\tau + (a^2/2)(1 + 3\cos 2\tau). \tag{19}$$

In order that w_1 shall be periodic the coefficient of $\cos\tau$ in the right member of (19) must be zero, otherwise the solution would contain the non-periodic or Poisson term $\tau\sin\tau$. On putting $\delta_1 = 0$, the general solution of (19) becomes

$$w_1 = A_1\cos\tau + B_1\sin\tau + (a^2/2)(1 - \cos 2\tau).$$

From the initial conditions (18) it follows that the constants of integration A_1 and B_1 are both zero, and the desired solution at this step becomes

$$\delta_1 = 0,\ w_1 = (a^2/2)(1 - \cos 2\tau).$$

Step 2: *Coefficient of* μ^2. The differential equation at this step is

$$\begin{aligned}\ddot{w}_2 + w_2 &= -\delta_2 w_0 + 4w_0 w_1 - 4w_0^3 + 2w_0\dot{w}_0^2 - 2\dot{w}_0\dot{w}_1\\ &= -a(\delta_2 + \tfrac{1}{2}a^2)\cos\tau - \tfrac{7}{2}a^3\cos 3\tau,\end{aligned}$$

and the desired solution is

$$\delta_2 = -\tfrac{1}{2}a^2, \ w_2 = \tfrac{7}{16}a^3(-\cos\tau + \cos 3\tau).$$

The remaining steps of the integration are entirely similar to the preceding step and by an induction to the general term it can readily be shown that the process can be carried on indefinitely. It will be observed that δ_j is zero if j is odd, and that each w_j carries the factor a^{j+1} and is a sum of cosines of multiples of τ having the opposite parity of j. The highest multiple of τ in w_j is $(j+1)$.

So far as the computation has been made, the solution of (15) is

$$\left.\begin{aligned} w &= a\cos\tau + (\mu a^2/2)(1-\cos 2\tau) - \frac{7\mu^2a^3}{16}(\cos\tau - \cos 3\tau) + \cdots, \\ \delta &= -\tfrac{1}{2}a^2\mu^2 + \cdots. \end{aligned}\right\} \tag{20}$$

The convergence of this solution will now be considered. In order to put the differential equation (15) in the same form as (11) of Macmillan's theorem, we let

$$w = W_1, \ dW_1/d\tau = W_2,$$

then (15) becomes

$$\begin{aligned} dW_1/d\tau &= f_1(W_1, W_2; \mu, \tau) = W_2, \\ dW_2/d\tau &= f_2(W_1, W_2; \mu, \tau) = -[(1+\delta)W_1 + W_2^2][1 - 2\mu W_1 + \cdots]. \end{aligned}$$

The f_1 and f_2 are obviously expansible as power series in W_1, W_2, and μ, and vanish for $W_1 = W_2 = \mu = 0$. The coefficients are constants and therefore satisfy the condition that they shall be uniform, continuous, and periodic functions of τ. Further, f_1 and f_2 converge for W_1, W_2, and μ sufficiently small numerically. Hence equations (15) satisfy the conditions of Macmillan's theorem and consequently the solution (20) converges for $|\mu|$ sufficiently small.

When (20) is substituted for w in (17) we obtain

$$z = c/2 + k\left[\mu a\cos\tau + \frac{\mu^2a^2}{2}(1-\cos 2\tau) - \frac{7\mu^3a^3}{16}(\cos\tau - \cos 3\tau) + \cdots\right].$$

As μ and a are both arbitrary and as they occur only in products as indicated, we may suppress either without loss of generality. Let us suppose $a = 1$. Then the periodic solution for the vertical motion of the particle becomes

$$\begin{aligned} z &= c/2 + k[\mu\cos\tau + (\mu^2/2)(1-\cos 2\tau) - (7\mu^3/16)(\cos\tau - \cos 3\tau) \\ &\qquad + \cdots], \\ \delta &= -\tfrac{1}{2}\mu^2 + (\ \ )\mu^4 + \cdots. \end{aligned} \tag{21}$$

In the expansion for z, the multiples of τ in the coefficient of μ^j have the same parity as j and therefore

$$z(\pi) = c/2 - k\mu = c/2 - \gamma.$$

Further

$$z(0) = c/2 + k\mu = c/2 + \gamma.$$

Hence the particle oscillates between the planes $z = c/2 + \gamma$ and $z = c/2 - \gamma$. The parameter γ is therefore a scale factor denoting the amplitude of the oscillation on either side of the plane $z = c/2$.

§6. *The Horizontal Motion.* When the solution (21) has been substituted in the first two equations of (8) and the transformation

$$t - t_0 = \tfrac{1}{2}\sqrt{(k/g)(1+\delta)}\,\tau, \quad \delta = -\tfrac{1}{2}\mu^2 + \cdots,$$

has been made, we obtain

$$\ddot{x} + k/p\,[\tfrac{1}{4} + \theta_1\mu + \theta_2\mu^2 + \cdots + \theta_j\mu^j + \cdots]\,x = 0, \tag{22}$$

and the same equation in y, where

$$\theta_1 = -\cos\tau, \ \theta_2 = -\tfrac{1}{8}(1 - \cos 2\tau),$$

and the remaining θ_j are likewise sums of cosines of multiples of τ having the same parity as j, the highest multiple being j.

Equation (22) is similar to the one first discussed by Hill* in 1877 in his celebrated memoir on the motion of the lunar perigee. A very complete list of references to the literature of the differential equations of this type is given by Baker on page 134 of his memoir "On Certain Linear Differential Equations of Astronomical Interest."†

Three cases arise in the solution of (22) depending upon the values of $(k/4p)$.‡ They are

Case I. $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and $\sqrt{k/p}$ not an integer.
Case II. $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and $\sqrt{k/p}$ an integer.
Case III. $\tfrac{1}{2}\sqrt{k/p} = 0$.

Since $k = p + c$ and since p and c are both positive in the physical problem under consideration, it follows that $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and Case III need not be considered.

---

* *The Collected Works of G. W. Hill*, Vol. I, pp. 243-270; *Acta Mathematica*, Vol. VIII, pp. 1-36.

† *Phil. Trans. of the Royal Society of London*, Series A, Vol. 216, pp. 129-186.

‡ Compare also Moulton's *Periodic Orbits*, Chap. III, § 52.

Case I.—This may be regarded as the general case.

The form of the solution of (22), first given by Floquet,* is

$$x = e^{i\alpha\tau}u, \tag{23}$$

where α and u are power series in μ, the former having constant coefficients, the latter periodic coefficients with the period 2π in τ. When (23) is substituted in (22) we obtain

$$\ddot{u} + 2i\alpha\dot{u} - \alpha^2 u + k/p\left[\tfrac{1}{4} + \theta_1\mu + \theta_2\mu^2 + \cdots + \theta_j\mu^j + \cdots\right]u = 0. \tag{24}$$

Now let

$$\begin{aligned} \alpha &= \tfrac{1}{2}\sqrt{k/p} + \alpha_1\mu + \alpha_2\mu^2 + \cdots, \\ u &= u_0 + u_1\mu + u_2\mu^2 + \cdots, \end{aligned} \tag{25}$$

be substituted in (24). We obtain a differential equation (24′), which is an identity in μ. Since the right number is zero, then the coefficient of each power of μ must also be zero. On solving the various differential equations thus obtained, we determine the u_j and α_j, the u_j as periodic functions of τ having the period 2π, and the α_j as constants so determined that the u_j shall be periodic.

Since the solution (23) is later multiplied by an arbitrary constant, see equations (39), we may choose $u(0) = 1$, from which it follows that

$$u_0(0) = 1, \; u_j(0) = 0, \; j = 1, \cdots \infty. \tag{26}$$

Step 0: *Coefficient* of μ^0. From the terms of (24′) that are independent of μ, we have the differential equation

$$\ddot{u}_0 + i\sqrt{k/p}\;\dot{u}_0 = 0,$$

and its solution is

$$u_0 = a_0 + b_0 e^{-i\sqrt{k/p}\,\tau}, \tag{27}$$

where a_0 and b_0 are the constants of integration. Since $\sqrt{k/p}$ is not an integer in this case, the term $e^{-i\sqrt{k/p}\tau}$ does not have the period 2π and we put $b_0 = 0$. From (26) it follows that $a_0 = 1$ and the desired solution at this step becomes $u_0 = 1$.

Step 1: *Coefficient of* μ. On equating to zero the terms in μ in (24′) we obtain

$$\ddot{u}_1 + i\sqrt{k/p}\;\dot{u}_1 = \sqrt{k/p}\;\alpha_1 + k/p\,\cos\tau. \tag{28}$$

* *Annales de l'École Normale Supérieure*, 1883-4.

The complementary function is

$$u_1 = a_1 + b_1 e^{-i\sqrt{k/p}\,\tau},$$

a_1 and b_1 being arbitrary constants.

It is well known in the theory of differential equations that the presence of any term in the right member which has *exactly* the same period as any term of the complementary function will yield Poisson terms in the particular integral, that is, terms containing τ outside of trigonometric or exponential symbols. Since the constant part of the right member of (28) has *exactly* the same period as the constant in the complementary function, then non-periodic terms will arise in the particular integral unless $a_1 = 0$. On putting $a_1 = 0$ the complete solution of (28) becomes

$$u_1 = a_1 + b_1 e^{-i\sqrt{k/p}\,\tau} + \frac{k}{k-p}[\cos\tau - i\sqrt{k/p}\sin\tau]. \tag{29}$$

From the periodicity and initial conditions it follows that

$$b_1 = 0,\ a_1 = \frac{k}{p-k}$$

and the desired solution at this step becomes

$$u_1 = \frac{k}{p-k}[1 - \cos\tau + i\sqrt{k/p}\sin\tau],\ a_1 = 0. \tag{30}$$

Step 2: Coefficient of μ^2. The differential equation at this step is

$$\begin{aligned} \ddot{u}_2 + i\sqrt{k/p}\,\dot{u}_2 &= a_2\sqrt{k/p} + \frac{k}{8p} - \frac{k^2}{2p(p-k)} \\ &+ \frac{k^2}{p(p-k)}(\cos\tau - \tfrac{1}{2}\cos 2\tau + \tfrac{1}{2}i\sqrt{k/p}\sin 2\tau) - (k/p)\cos 2\tau\ , \end{aligned} \tag{31}$$

and the solution which satisfies the periodicity and initial conditions is

$$\left.\begin{aligned} u_2 &= \frac{7pk^3 - 22p^2k^2 + 4p^3k - k^4}{4p(p-k)^2(k-4p)} - \frac{k^2}{(p-k)^2}(\cos\tau - i\sqrt{k/p}\sin\tau) \\ &+ \frac{2pk^2 - 4p^2k - k^3}{4p(p-k)(k-4p)}\cos 2\tau + i\sqrt{k/p}\,\frac{k(k+2p)}{4(p-k)(k-4p)}\sin 2\tau, \\ a_2 &= \tfrac{1}{8}\sqrt{k/p}\left(\frac{5k-p}{p-k}\right). \end{aligned}\right\} \tag{32}$$

So far as the computation has been carried out it is observed that the

a_j are real constants and that the u_j consist of sums of cosines and i times sines of multiples of τ, the highest multiple being j. It will now be shown by induction that these properties hold in general.

Let us suppose that $a_1, \cdots, a_{n-1}, u_0, u_1, \cdots, u_{n-1}$ have been computed and that the various a_j are real and that the u_j, $j=1, \cdots, n-1$, have the form

$$u_j = \sum_{l=0}^{j} (A_j^{(l)} \cos l\tau + iB_j^{(l)} \sin l\tau), \tag{33}$$

where the $A_j^{(l)}$ and $B_j^{(l)}$ are real constants. We wish to show that a_n can be determined as a real constant and that u_n has the same form as (33) when $j=n$.

Step n: Coefficient of μ^n. The differential equation at this step is

$$\ddot{u}_n + i\sqrt{k/p}\,\dot{u}_n = a_n\sqrt{k/p} + U_n(a_1, \cdots, a_{n-1}; \dot{u}_1, \cdots, u_{n-1}). \tag{34}$$

The only undetermined constant which enters (34) is a_n and it is written explicitly in so far as it occurs. The function U_n is linear in $u_1, \cdots, u_{n-1}$. The terms of U_n which arise from $2ia\dot{u}$ in (24) have the form

$$2ia_l\dot{u}_{n-l}, \quad l=1, \cdots, n-1,$$

and are cosines and i times sines of multiples of τ, the highest multiple being $n-1$. The terms which arise from a^2u in (24) have the form

$$-(\ \ )a_l\, a_m\, u_{n-(l+m)}, \quad l, m = 0, \cdots, n-1, \quad l+m \leqq n-1;$$

where ( ) denotes 1 if $l=m$ and 2 if $l \neq m$. These terms have the same form as the preceding terms. When $l=m=0$, the term $-a_0{}^2u_n$ will cancel off with the term $(k/4p)u_n$ arising from the last factor of (24) and therefore the highest multiple of τ in the sines and cosines is $n-1$. The last factor of (24) gives, in addition to the term $(k/4p)u_n$ just considered, terms of the type

$$(k/p)\ \theta_l u_{n-l}, \quad l=1, \cdots, n-1.$$

These terms are also of the same form as (33) but the highest multiple of τ which they yield is n. Therefore

$$U_n = \sum_{l=0}^{n} [a_l^{(n)} \cos l\tau + ib_l^{(n)} \sin l\tau], \tag{35}$$

where $a_l^{(n)}$ and $b_l^{(n)}$ are real constants.

Let us now determine the periodic solution of (34). In order that the

solution shall be periodic, the right member of (34) must contain no constant terms. Hence

$$a_n = -\sqrt{p/k}\, a_0^{(n)},$$

a real constant. The complete solution of (34) then becomes

$$u_n = a_n + b_n e^{-i\sqrt{k/p}\,\tau} + \bar{U}_n,$$

where a_n and b_n are the constants of integration, and the particular integral $\bar{U}_n$ has the same form as (35), except that it has no constant term. From the periodicity and the initial conditions we have

$$b_n = 0, \; a_n = -\bar{U}_n(0),$$

and the desired solution for u_n is the same as when $j = n$. Hence the properties of a_j and u_j already stated hold in general.

A second solution of (22) could be constructed in an entirely similar way but it is not necessary to make the construction as this solution can be obtained directly from the former solution.

Let the solution of (24) already obtained be denoted by $u(\tau, +i)$. Then one solution of (22) is

$$x = e^{+ia\tau}\, u(\tau, +i). \tag{36}$$

Since the differential equation (22) is independent of i, a change in the sign of i in (36) will still give a solution, viz.,

$$x = e^{-ia\tau}\, u(\tau, -i). \tag{37}$$

Thus a second solution can be obtained by changing the sign of i in the first solution.

The determinant formed by the two solutions (36) and (37) together with their derivatives with respect to τ is a constant,* and its value can be computed most readily when $\tau = 0$. This determinant is

$$D = -2[ia + \dot{u}(0, +i)] = -i\sqrt{k/p} + \text{terms in } \mu, \tag{38}$$

which is different from zero for $\mu = 0$ and therefore remains different from zero for $|\mu|$ sufficiently small. The two solutions (36) and (37) therefore constitute a fundamental set and the most general solution of (22), as well as of the similar equation in y, is

* Moulton's *Periodic Orbits*, § 18.

$$\left.\begin{aligned}
x &= A_1 e^{ia\tau} u^{(1)} + A_2 e^{-ia\tau} u^{(2)},\\
y &= B_1 e^{ia\tau} u^{(1)} + B_2 e^{-ia\tau} u^{(2)},\\
a &= \tfrac{1}{2}\sqrt{k/p}\left[1 + \frac{5k-p}{4(p-k)}\mu^2 + \cdots\right],\\
u^{(1)} &= u(\tau, +i) = 1 + \frac{k}{k-p}\left[1 - \cos\tau + i\sqrt{k/p}\sin\tau\right]\mu\\
&\quad + \left[\frac{7pk^3 - 22p^2k^2 + 4p^3k - k^4}{4p(p-k)^2(k-4p)} - \frac{k^2}{(p-k)^2}(\cos\tau - i\sqrt{k/p}\sin\tau)\right.\\
&\quad + \frac{2pk^2 - 4p^2k - k^3}{4p(p-k)(k-4p)}\cos 2\tau\\
&\quad \left. + i\sqrt{k/p}\,\frac{k(k+2p)}{4(p-k)(k-4p)}\sin 2\tau)\right]\mu^2 + \cdots,\\
u^{(2)}(+i) &= u^{(1)}(-i),
\end{aligned}\right\} \quad (39)$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

By using Macmillan's theorem, quoted in § 3, it can readily be shown that the above solutions converge for all τ in the interval $0 \leqq \tau \leqq 2\pi$ provided $|\mu|$ is sufficiently small.

Case II. Suppose $\sqrt{k/p} = \nu$, an integer. Since $k = p + c$ and p and c are both positive, then $\nu > 1$. Two sub-cases arise depending upon the value of ν. They are:

Sub-case I. $\nu = 2$,
Sub-case II. $\nu \neq 2$.

Sub-case I. The construction proceeds as in Case I until equation (27) is reached, and this becomes

$$u_0 = a_0 + b_0 e^{-2i\tau}, \qquad (40)$$

where both terms have the period 2π. From the initial condition $u_0(0) = 1$ it follows that $b_0 = 1 - a_0$, and the solution for u_0 becomes

$$u_0 = a_0 + (1 - a_0)e^{-2i\tau}, \qquad (41)$$

with the constant a_0 remaining arbitrary at this step.

Step 1: *Coefficient of* μ. When we equate to zero the coefficient of μ in (24″), that is in equation (24) after (25) has been substituted and $\sqrt{k/p}$ replaced by 2, we obtain

$$\begin{aligned}\ddot{u}_1 + 2i\dot{u} = 2a_1a_0 - 2a_1(1-a_0)e^{-2i\tau} + 2a_0e^{i\tau}\\ + 2e^{-i\tau} + 2(1-a_0)e^{-3i\tau}. \qquad (42)\end{aligned}$$

The complementary function of this equation is

$$u_1 = a_1 + b_1 e^{-2i\tau}.$$

Since terms of *exactly* the same period as those of the complementary function occur in the right member of (42), viz., constants and terms in $e^{-2i\tau}$, the particular integral will contain non-periodic terms unless the constants and the coefficient of $e^{-2i\tau}$ are put equal to zero. Since we seek a periodic solution we therefore put

$$2a_1 a_0 = 0,\ 2a_1(1 - a_0) = 0. \tag{43}$$

These equations are satisfied by $a_1 = 0$, a_0 arbitrary, but a_0 must be different from 1 or this case would be the same as Case I. The complete solution of (42) then becomes

$$u_1 = a_1 + b_1 e^{-2i\tau} - \tfrac{2}{3} a_0 e^{i\tau} + 2e^{-i\tau} - \tfrac{2}{3}(1 - a_0) e^{-3i\tau}. \tag{44}$$

Since $u_1(0) = 0$, then

$$b_1 = -(\tfrac{4}{3} + a_0).$$

At this step a_0 and a_1 still remain undetermined.

Step 2: Coefficient of μ^2. The terms of (24″) which contain the factor μ^2 give the equation

$$\begin{aligned}\ddot{u}_2 + 2i\dot{u}_2 = {} & [2a_2 a_0 + \tfrac{7}{6} a_0 + 2] + [-2a_2(1 - a_0) - \tfrac{7}{6} a_0 + \tfrac{19}{6}] e^{-2i\tau} \\ & + 2a_1 e^{i\tau} - \tfrac{10}{3} a_0 e^{2i\tau} - \tfrac{8}{3} e^{-i\tau} \\ & - 2(a_1 + \tfrac{4}{3}) e^{-3i\tau} - \tfrac{10}{3}(1 - a_0) e^{-4i\tau}.\end{aligned}$$

On equating to zero the constants and the coefficient of $e^{-2i\tau}$ in the right member, as at the preceding step, we have

$$\left.\begin{aligned}2a_2 a_0 - \tfrac{7}{6} a_0 + 2 = 0, \\ -2a_2(1 - a_0) - \tfrac{7}{6} a_0 + \tfrac{19}{6} = 0.\end{aligned}\right\} \tag{45}$$

These equations are satisfied by

$$a_2 = \pm \tfrac{1}{12}\sqrt{217},\quad a_0 = \tfrac{1}{2} \mp \tfrac{1}{14}\sqrt{217}, \tag{46}$$

where the upper signs are to be taken together, also the lower. When equations (46) are satisfied, the general solution for u_2 will be periodic, being

$$\begin{aligned}u_2 = {} & a_2 + b_2 e^{-2i\tau} - \tfrac{2}{3} a_1 e^{i\tau} + \tfrac{5}{12} a_0 e^{2i\tau} - \tfrac{8}{3} a_0 e^{-i\tau} \\ & + \tfrac{2}{3}(a_1 + \tfrac{4}{3}) e^{-3i\tau} + \tfrac{5}{12}(1 - a_0) e^{-4i\tau}.\end{aligned}$$

Since $u_2(0) = 0$ then

$$b_2 = -a_2 + \tfrac{8}{3}a_0 - \tfrac{47}{36}.$$

The undetermined constants at this step are a_2 and a_1.

Step 3: Coefficient of μ^3. The differential equation at this step is

$$\begin{aligned}\ddot{u}_3 + 2i\dot{u}_3 = U_3 = &-2ia_3\dot{u}_0 - 2ia_2\dot{u}_1 + 2a_3u_0 + 2a_2u_1 \\ &- u_0\theta_3 + u_1(\tfrac{1}{2} - 4\cos 2\tau) + 4u_2\cos\tau.\end{aligned}$$

If we consider only the part of U_3 which involves the arbitrary constants that are determined at this step we have

$$\begin{aligned}U_3 = &\ 2a_3a_0 + a_1(2a_2 + \tfrac{7}{6}) + d_1 \\ &+ [-2(1-a_0)a_3 + a_1(2a_2 - \tfrac{7}{6}) + d_2e^{-2i\tau}] \\ &+ \text{terms in } e^{\pm i\tau},\ e^{2i\tau},\ e^{\pm 3i\tau},\ e^{\pm 5i\tau},\ e^{-4i\tau},\end{aligned}\tag{47}$$

where d_1 and d_2 are known constants. To make u_3 periodic we equate to zero the constants and the coefficient of $e^{-2i\tau}$ in (47) giving

$$\left.\begin{aligned}2a_3a_0 + a_1(2a_2 + \tfrac{7}{6}) + d_1 = 0, \\ -2(1-a_0)a_3 + a_1(2a_2 - \tfrac{7}{6}) + d_2 = 0.\end{aligned}\right\}\tag{48}$$

The determinant of the coefficients of a_3 and a_1 in (48) is

$$D_1 = \pm \tfrac{2}{3}\sqrt{217},\tag{49}$$

where the upper sign here is to be taken with the upper signs in (46), similarly for the lower signs. Since this determinant D_1 is different from zero, equations (48) can be solved for a_3 and a_1, and the solutions are unique for either set of values in (46). When (48) is satisfied, the solution for u_3 will therefore be periodic. Two constants of integration, a_3 and b_3, say, will arise from the complementary function. One constant, b_3, say, will be determined from the initial condition $u_3(0) = 0$, while a_2 and a_3 remain undetermined at this step.

The succeeding steps of the integration are entirely similar to step 3 just considered. So far as the constants of integration are concerned one of them, b_j say, is determined by the initial condition $u_j(0) = 0$ at the step where it arises. The other constant a_j of the step j, is not determined until the step $j + 2$ is reached, where, on equating to zero the constants and the coefficient of $e^{-2i\tau}$ in the right member of the differential equation in u_{j+2}, two linear equations in a_{j+2} and a_j are obtained and the determinant of their coefficients is the same as D_1 in (49). Hence a_{j+2} and a_j can be uniquely determined provided one set of values is taken in (46).

It is thus evident that two solutions of (24″) can be obtained in the foregoing construction, according as the upper or lower signs are taken in (46). The corresponding solutions of (22) are

$$x = x_1 = e^{i(1+{}^1/_{12}\sqrt{217}\,\mu^2+\cdots)\tau}\,[(½ - {}^1/_{14}\sqrt{217}) + (½ + {}^1/_{14}\sqrt{217})e^{-2i\tau} + (\;)\mu + \cdots],$$

and

$$x = x_2 = e^{i(1-{}^1/_{12}\sqrt{217}\,\mu^2+\cdots)\tau}\,[(½ + {}^1/_{14}\sqrt{217}) + (½ - {}^1/_{14}\sqrt{217})e^{-2i\tau} + (\;)\mu + \cdots].$$

If we proceed as in Case I to obtain two solutions by changing the sign of i in the preceding solutions it would appear that two additional solutions could be obtained, but this is impossible since the differential equation (22) is only of the second order and admits of but two solutions. This apparent difficulty is overcome if the factor $e^{i\tau}$ is multiplied into the part of the solutions contained in the square brackets [ ]. Thus

$$\left.\begin{aligned} x_1 &= e^{i({}^1/_{12}\sqrt{217}\,\mu^2+\cdots)\tau}\,[(½ - {}^1/_{14}\sqrt{217})e^{i\tau} + \\ &\quad + (½ + {}^1/_{14}\sqrt{217})e^{-i\tau} + (\;)\mu + \cdots], \\ x_2 &= e^{-i({}^1/_{12}\sqrt{217}\,\mu^2+\cdots)\tau}\,[(½ + {}^1/_{14}\sqrt{217})e^{i\tau} + \\ &\quad + (½ - {}^1/_{14}\sqrt{217})e^{-i\tau} + (\;)\mu + \cdots]. \end{aligned}\right\} \tag{50}$$

It is obvious, therefore, that the two solutions x_1 and x_2 differ only in the sign of i or of $\sqrt{217}$.

If the exponentials in the square brackets of (50) are expressed in trigonometric form, these solutions become

$$\left.\begin{aligned} & x_1 = e^{ia\tau}u^{(1)},\; x_2 = e^{-ia\tau}u^{(2)}, \\ & a = {}^1/_{12}\sqrt{217}\,\mu^2 + \cdots, \\ & u^{(1)} = \cos\tau - i\,{}^1/_7\sqrt{217}\sin\tau + (\;)\mu + \cdots, \\ & u^{(2)}(+i) = u^{(1)}(-i). \end{aligned}\right\} \tag{51}$$

The terms represented by $u^{(1)}$ and $u^{(2)}$ differ not only in i but also in the sign of $\sqrt{217}$. They are power series in μ with sums of cosines and $i\sqrt{217}$ times sines of multiples of τ, the highest multiple in the coefficient of μ^j being $j+1$.

The determinant of the two solutions in (51) together with their derivatives is a constant, as in Case I, and its value at $\tau = 0$ is

$$D_2 = -2[ia + \dot{u}^{(1)}(0)] = i\,{}^2/_7\sqrt{217} + (\;)\mu + \cdots,$$

which is different from zero for $\mu = 0$, and therefore remains different from zero for $|\mu|$ sufficiently small. Hence the two solutions (51) constitute a fundamental set, and the most general solutions of (22) and the similar equation in y are

$$x = A_1 e^{i\alpha\tau} u^{(1)} + A_2 e^{-i\alpha\tau} u^{(2)}, \quad y = B_1 e^{i\alpha\tau} u^{(1)} + B_2 e^{-i\alpha\tau} u^{(2)}, \tag{52}$$

where α, $u^{(1)}$, and $u^{(2)}$ are defined in (51), and A_1, A_2, B_1, B_2 are the constants of integration.

Sub-case II. When ν is an integer different from 2, the construction is the same as in the preceding sub-case until equations (43) are reached. The equations analogous to (43) are

$$\nu a_1 a_0 = 0, \qquad \nu a_1 (1 - a_0) = 0. \tag{53}$$

These are satisfied by $a_1 = 0$, a_0 arbitrary, but a_0 must be different from 1 as in the preceding sub-case.

The equations which correspond to (45) are

$$\left.\begin{aligned} a_0 \left\{ \nu a_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} &= 0, \\ (1 - a_0) \left\{ -\nu a_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} &= 0. \end{aligned}\right\} \tag{54}$$

Since $a_0 \neq 1$ in Case II, then the solutions of (54) are

$$a_0 = 0, \ a_2 = \frac{\nu(5\nu^2 - 1)}{8(\nu^2 - 1)}. \tag{55}$$

The equations analogous to (48) are

$$a_1 \left\{ \nu a_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} = 0, \qquad -\nu a_3 + \nu^2 \theta_3^{(0)} = 0, \tag{56}$$

where $\theta_3^{(0)}$ is the constant part of θ_3 in equation (22). The solutions of these equations are

$$a_1 = 0, \ a_3 = \nu \theta_3^{(0)}.$$

The constants a_2 and a_4 are determined in the same way at the next step as a_1 and a_3 were found in the preceding step. They have the same coefficient in the equations analogous to (56) as a_1 and a_3 have in (56), and similarly for the succeeding steps. It will be found that

$$a_j = 0, \ j = 0, 1, \cdots, \nu - 3,$$

and that the remaining a_j are, in general, different from zero.

So far as the computation has been made we find

$$\left.\begin{aligned}
&u_0 = e^{-i\nu\tau},\\
&u_1 = \frac{\nu^2}{\nu^2-1}e^{-i\nu\tau} + \nu^2/2\left[\frac{e^{-i(\nu-1)\tau}}{\nu-1} - \frac{e^{-i(\nu+1)\tau}}{\nu+1}\right],\\
&u_2 = \left[\frac{\nu^4}{(\nu^2-1)^2} + \frac{\nu^2}{8(\nu^2-4)}\right]e^{-i\nu\tau} - \frac{\nu^4}{2(\nu^2-1)}\left[\frac{e^{-i(\nu-1)\tau}}{\nu-1}\right.\\
&\qquad\left.-\frac{e^{-i(\nu+1)\tau}}{\nu+1}\right] - \nu^2/32\left[\frac{e^{-i(\nu-2)\tau}}{\nu-2} - \frac{e^{-i(\nu+2)\tau}}{\nu+2}\right],\\
&a_1 = 0,\ a_2 = \frac{\nu(5\nu^2-1)}{8(\nu^2-1}.
\end{aligned}\right\}$$

When these terms are substituted in (25) and the result in (23) we obtain one solution of (22). Let it be denoted by

$$x = e^{ia\tau}u^{(1)}. \tag{58}$$

Another solution can be obtained by changing the sign of i in (58), thus

$$x = e^{-ia\tau}u^{(2)}, \tag{59}$$

where
$$u^{(2)}(+i) = u^{(1)}(-i).$$

The determinant of these two solutions and their first derivatives is

$$-2[ia + \dot{u}^{(1)}(0)] = i\nu + (\quad)\mu + \cdots,$$

which is different from zero for $|\mu|$ sufficiently small. The solutions (58) and (59) therefore constitute a fundamental set, and the general solutions of (22) and the corresponding equation in y are of the same form as (39) or (52), viz.,

$$x = A_1e^{ia\tau}u^{(1)} + A_2e^{-ia\tau}u^{(2)},\ y = B_1e^{ia\tau}u^{(1)} + B_2e^{-ia\tau}u^{(2)}, \tag{60}$$

where a and $u^{(1)}$ have the values in (58). If the exponentials in (58) are expressed in trigonometric form then

$$u^{(1)} = \sum_{j=0}^{\infty}\sum_{l=0}^{j}[A^{(j)} + B^{(j)}_{\pm l}\cos(\nu\pm l)\tau + iC^{(j)}_{\pm l}\sin(\nu\pm l)\tau]\mu^j, \tag{61}$$

where $A^{(j)}$, $B^{(j)}_{\pm l}$, and $C^{(j)}_{\pm l}$ are real constants. In particular

$$A^{(j)} = 0,\ j = 0, 1, 2,\ B^{(j)}_{\pm l} = -C^{(j)}_{\pm l},\ j = 0, 1, 2,\ l = 0, 1, 2,$$

$$B^{(0)}_0 = 1,\ B^{(1)}_0 = -\frac{\nu^2}{\nu^2-1},$$

$$B^{(1)}_{-1} = \frac{\nu^2}{2(\nu-1)},\ B^{(1)}_{-1} = -\frac{\nu^2}{2(\nu+1)},$$

$$B^{(2)}_0 = \frac{\nu^4}{(\nu^2-1)^2} + \frac{\nu^4}{8(\nu^2-4)},\ B^{(2)}_{-1} = -\frac{\nu^4}{2(\nu-1)(\nu^2-1)},$$

$$B^{(2)}_{+1} = -\frac{\nu^4}{2(\nu+1)(\nu^2-1)},\ B^{(2)}_{-2} = -\frac{\nu^2}{32(\nu-2)},$$

$$B^{(2)}_{+2} = \frac{\nu^2}{32(\nu+2)}.$$

From (57) it follows that

$$A^{(j)} = 0,\ j = 0, 1, \cdots, \nu - 3.$$

The remaining $A^{(j)}$ are different from zero, in general.

If the factors $e^{i(\nu/2)\tau}$ and $e^{-i(\nu/2)\tau}$ are taken with $u^{(1)}$ and $u^{(2)}$, respectively, in (60), then $u^{(1)}$ and $u^{(2)}$ will have cosines and sines of $(\nu/2 \pm l)\tau$ instead of $(\nu \pm l)\tau$ as in (61).

§ 7. *The Arbitrary Constants.* Each of the solutions for x, y, and z contains two arbitrary constants, and the values of these constants for the physical problem will now be considered.

In this discussion no discrimination will be made between the two cases of the preceding section. It was for this reason that the same notation was chosen for the solutions (39), (52), and (60).

Besides satisfying the differential equations (8), the solutions for x, y, and z must also satisfy (7), viz.,

$$x^2 + y^2 - 2pz = 0, \tag{7}$$

and their derivatives with respect to t must satisfy the *vis viva* integral (4). When the transformation

$$t - t_0 = \tfrac{1}{2}\sqrt{k/g}(1+\delta)\tau,\ \delta = -\tfrac{1}{4}\mu^2 + \cdots$$

is made in (4), this integral becomes

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (k/2)(1+\delta)(c-z). \tag{62}$$

Since any horizontal section of the surface (7) is a circle with centre on the z-axis, the xy-axes may be rotated about the z-axis without changing the form of (7), and we may therefore suppose that the particle lies in the xz-plane at the initial time, or $y = 0$ at $\tau = 0$. Since $u^{(1)}(0) = u^{(2)}(0) = 1$, then it follows that $B_1 + B_2 = 0$.

As $\dot{z}(0) = 0$, the particle must be initially projected in a horizontal plane. Since any horizontal section of the surface of constraint is a circle with centre on the z-axis and since $u(0) = 0$, then it follows that $\dot{x}(0) = 0$. When this condition is imposed on the solution for x, we have

$$[ia+\dot{u}^{(1)}(0)]\ [A_1-A_2]=0.$$

The factor $[ia+\dot{u}^{(1)}(0)]$ is different from zero for $|\mu|$ sufficiently small and therefore

$$A_1=A_2.$$

Hence the solutions for x and y become

$$x=A[e^{ia\tau}u^{(1)}+e^{-ia\tau}u^{(2)}],\quad y=B[e^{ia\tau}u^{(1)}-e^{-ia\tau}u^{(2)}],\tag{63}$$

where the now unnecessary subscripts on the constants A and B have been dropped.

On putting $\tau=0$ in (63) and (21) we get

$$\left.\begin{array}{ll} x(0)=2A, & \dot{x}(0)=0,\\ y(0)=0, & \dot{y}(0)=2B[ia+\dot{u}^{(1)}(0)],\\ z(0)=c/2+k\mu, & \dot{z}(0)=0. \end{array}\right\}\tag{64}$$

When these values are substituted in (7) and (62) we find

$$A=\pm\tfrac{1}{2}\sqrt{p(c+2k\mu)}=\pm\tfrac{1}{2}\sqrt{p(c+2\gamma)},$$
$$B=\pm\frac{\sqrt{k(1+\delta)(c-2\gamma)}}{4[ia+\dot{u}^{(1)}(0)]}=\mp i\frac{\sqrt{k(1+\delta)(c-2\gamma)}}{4[a-i\dot{u}^{(1)}(0)]}.$$

Thus A is real and B is purely imaginary. Hence the solutions of equations (8), in terms of τ, are

$$\left.\begin{array}{l} x=\pm\tfrac{1}{2}\sqrt{p(c+2\gamma)}\ [e^{ia\tau}u^{(1)}+e^{-ia\tau}u^{(2)}],\\ y=\mp\dfrac{i\sqrt{k(1+\delta)(c-2\gamma)}}{4[a-i\dot{u}^{(1)}(0)]}\ [e^{ia\tau}u^{(1)}-e^{-ia\tau}u^{(2)}],\\ z=c/2+k[\mu\cos\tau+\tfrac{1}{2}\mu^2(1-\cos 2\tau)\\ \qquad\qquad -\tfrac{7}{16}\mu^3(\cos\tau-\cos 3\tau)+\cdots],\\ \delta=-\tfrac{1}{2}\mu^2+\cdots,\\ \mu=\gamma/k,\qquad\qquad k=p+c. \end{array}\right\}\tag{65}$$

The terms a, $u^{(1)}$, and $u^{(2)}$ are defined in (39), (52), and (60) according to the value of $\sqrt{k/p}$. The double signs in (65) depend upon the octants of space into which the particle is initially projected.

If the exponentials in (65) are expressed in trigonometric form, the solutions of (22) for x and y become

$$x=\sqrt{p(c+2\gamma)}\ [\{\ \}_1\cos a\tau-\{\ \}_2\sin a\tau],$$
$$y=\frac{\sqrt{k(1+\delta)(c-2\gamma)}}{2[a-i\dot{u}^{(1)}(0)]}\ [\{\ \}_1\sin a\tau+\{\ \}_2\cos a\tau],$$

where $\{\ \}_1$ and $\{\ \}_2$ are power series in μ which have different values for the different cases of § 6. For Case I,

$$\{\ \}_1=1+\sum_{j=1}^{\infty}\sum_{l=0}^{j}[a_{jl}\cos l\tau]\mu^j,$$

$$\{\ \}_2=\sum_{j=1}^{\infty}\sum_{l=1}^{j}[b_{jl}\sin l\tau]\mu^j,$$

where a_{jl} and b_{jl} are the coefficients of $\cos l\tau$ and $\sin l\tau$, respectively, in u_j, equations (30), (32), and (33).

For Case II,

$$\{\ \}_1=\sum_{j=0}^{\infty}\sum_{l=0}^{j}[A^{(j)}+B^{(j)}_{\pm l}\cos(\nu\pm l)\tau]\mu^j,$$

$$\{\ \}_2=\sum_{j=1}^{\infty}\sum_{l=0}^{j}[C^{(j)}_{\pm l}\sin(\nu\pm l)\tau]\mu^j.$$

The constants $A^{(j)}$, $B^{(j)}_{\pm l}$, and $C^{(j)}_{\pm l}$ have the same values as in (61) for sub-case II, and for sub-case I they are, in so far as the computation has been carried out,

$$A^{(0)}=0,\ B^{(0)}_0=1,\ C^{(0)}_0=-\tfrac{1}{7}\sqrt{217}.$$

There still remain in the solutions (65) two constants which have not been determined. They are c and γ $(=k\mu)$. Since c is the constant of integration arising in the *vis viva* integral, its value depends upon the initial velocity of the particle. As $x'=z'=0$ at $t=t_0$, the initial velocity may be denoted by y'_0. Then from the *vis viva* integral (4) it follows that

$$y'^2_0=2g(c-z_0)=g(c-2\gamma),\ \text{or}\ c-2\gamma=y'^2_0/g.$$

Since, further, $c+2\gamma=2z_0$, then

$$c=z_0+y'^2_0/2g,\ \ \gamma=\tfrac{1}{2}(z_0-y'^2_0/2g).$$

Thus the constants c and γ are functions of the initial velocity and the initial height.

Let us now return to the three cases of § 3. Since $z_0=c/2+\gamma$ these cases become:

Case I. $0<\gamma\leqq c/2$,
Case II. $\gamma=0$,
Case III. $\gamma<0$, $|\gamma|\leqq c/2$.

In Case I and Case III the particle oscillates between the horizontal planes $z=c/2+\gamma$ and $z=c/2-\gamma$. The two orbits are geometrically the same but the motion in the one orbit is half a period ahead of the motion in the

other. If $\gamma = c/2$, then the initial velocity is zero and as there is no lateral projection the particle will move in the vertical parabola $x^2 = 2pz$. If $\gamma = -c/2$, then $z_0 = 0$, and the particle is projected from the lowest point with the initial velocity $y'_0 = \sqrt{2gc}$. It will therefore move in the vertical parabola $y^2 = 2pz$ which is dynamically the same orbit as $x^2 = 2pz$. The orbits when $\gamma = \pm c/2$ correspond therefore to the simple pendulum.

In Case II when $\gamma = 0$, then $z = c/2$, a constant, and the differential equations of motion become

$$x'' + (g/p)x = 0,\ y'' + (g/p)y = 0,\ z'' = 0.$$

Their periodic solutions which satisfy the initial conditions are

$$x = \sqrt{pc}\,\cos\sqrt{g/p}\,t, \quad y = \sqrt{pc}\,\sin\sqrt{g/p}\,t, \quad z = c/2.$$

In this case the particle moves in a circle the plane of which is parallel to the xy-plane and at a distance $c/2$ above it.

In the first paragraph of § 5 it is stated that the inequality $z < p/2 + c$ must hold in order that the expression in (13) will converge. Now the maximum value of z is

$$z = c/2 + |\gamma|,$$

and therefore the above inequality becomes

$$c/2 + |\gamma| < p/2 + c,$$

or

$$|\gamma| < \tfrac{1}{2}(p + c).$$

Since $|\gamma|$ must not exceed $c/2$ in order that the initial velocity shall be real, it follows that the inequality $z < p/2 + c$ will always be satisfied for real initial conditions.

(B). PERIODIC ORBITS ON A SURFACE OF REVOLUTION.

§ 8. *The Method of Solution.*—Let us next consider the construction of the periodic orbits described on the more general surface of revolution represented by the equation

$$F(x, y, z) = x^2 + y^2 - 2pz + 2\epsilon f(z) = 0. \tag{2}$$

The differential equations of motion have already been found, equations (3) and (5). The method of constructing the periodic solution of these equations is to first make the analytic continuation with respect to ϵ of the periodic solution for the vertical motion obtained in (A) where $\epsilon = 0$, and then substitute this solution for z in the first two equations in (3). We thus

obtain two differential equations having periodic coefficients somewhat similar to (22).

§ 9. *The Equation of Variation.*—Let us substitute in the last equation of (3)

$$\left.\begin{array}{c} z = \bar{z} + \xi,\ t - t_0 = \tfrac{1}{2}\sqrt{k/g(1+\delta)}\,\tau, \\ \delta = -\tfrac{1}{2}\mu^2 + \cdots, \end{array}\right\} \tag{66}$$

where $\bar{z}$ denotes the solution obtained in (21), and ξ is a function of τ which vanishes with ϵ. We obtain

$$\ddot{\xi} + \Theta_1\dot{\xi} + \Theta_2\xi = Z_0 + \epsilon Z_1 + \cdots + \epsilon^j Z_j + \cdots, \tag{67}$$

where the undefined terms have the following properties:

(1). The functions Θ_1 and Θ_2 are periodic functions of τ, but we shall show that it is not necessary to know the explicit values of these functions in order to solve (67).

(2). The functions $Z_0, Z_1, \cdots, Z_j$ denote power series in ξ having coefficients which are power series in μ with sums of cosines of multiples of τ in their coefficients. These functions also contain additional terms in $\dot{\xi}$ and $\dot{\xi}^2$, the former being multiplied by power series in μ with sums of sines of multiples of τ in the coefficients, the latter by similar series except that they contain cosines. The function Z_0 contains no linear terms in ξ or $\dot{\xi}$.

If we neglect the right member of (67) we obtain

$$\ddot{\xi} + \Theta_1\dot{\xi} + \Theta_2\xi = 0, \tag{68}$$

which is called *the equation of variation.* The generating solution is $z = \bar{z}$, or the expression for z in (21).

Now it has been shown by Poincaré* that if the generating solution contains an arbitrary constant which does not occur in the original differential equations of motion, viz., equations (3), then a solution of the equations of variation can be obtained by differentiating the generating solution with respect to this constant. Three constants occur in (21), viz., c, μ, and t_0, the latter entering implicitly through τ. The constant c occurs in 2λ, equation (5), and therefore enters the differential equations (3). The two remaining constants, μ and t_0, do not occur in (3) and therefore may be used in applying Poincaré's theorem. Each constant yields a solution of (68) and since the differential equation is only of the second order, both its solutions may be obtained from the generating solution. Hence it is not necessary to know the coefficients Θ_1 and Θ_2 in (68) in order to solve the differential equation.

---

* *Les Méthodes Nouvelles de la Mécanique Celeste,* Vol. I, chap. IV. *Loc. cit.*

Consider, first, the constant t_0. Then one solution of (68) is

$$\xi = \partial \bar{z}/\partial t_0 = -2\sqrt{\frac{g}{k(1+\delta)}}\,\partial \bar{z}/\partial t = 2\sqrt{\frac{g}{k(1+\delta)}}\,\mu S(\tau), \tag{69}$$

where

$$S(\tau) = \sin\tau - \mu\sin 2\tau - \tfrac{7}{16}\mu^2(\sin\tau - 3\sin 3\tau) + \cdots. \tag{70}$$

Since this solution is later multiplied by an arbitrary constant, see equation (76), we may drop the constant factor of $S(\tau)$ in (69) and take

$$\xi = S(\tau) \tag{71}$$

as the solution.

Considering the constant μ, we obtain as the second solution of (68)

$$\xi = \frac{\partial \bar{z}}{\partial \mu} = \left(\frac{\partial \bar{z}}{\partial \mu}\right) + \frac{\partial \bar{z}}{\partial \tau}\frac{\partial \tau}{\partial \delta}\frac{\partial \delta}{\partial \mu}, \tag{72}$$

where the parentheses ( ) denote differentiation in so far as μ occurs explicitly in $\bar{z}$. Now

$$\left.\begin{aligned} \frac{\partial \bar{z}}{\partial \mu} &= kC(\tau) = k[\cos\tau + \mu(1-\cos 2\tau) \\ &\qquad\qquad - \frac{21\mu^2}{16}(\cos\tau - \cos 3\tau) + \cdots], \\ \frac{\partial \tau}{\partial \delta} &= -\tfrac{1}{2}\frac{\tau}{1+\delta}, \quad \frac{\partial \mu}{\partial \delta} = \mu[-1 + \text{power series in } \mu^2]. \end{aligned}\right\} \tag{73}$$

Therefore the solution (72) becomes

$$\xi = k[C(\tau) + \tau K S(\tau)],$$

where

$$K = \mu^2[-1 + \text{power series in } \mu^2].$$

As in the previous solution we may drop the constant factor k and take

$$\xi = C(\tau) + \tau K S(\tau), \tag{74}$$

as the second solution of (68).

The two solutions (71) and (74) constitute a fundamental set, since the determinant of these two solutions together with their derivatives is different from zero for $|\mu|$ sufficiently small, being

$$\Delta = -1 + \text{power series in } \mu. \tag{75}$$

Hence the general solution of (68) is

$$\xi = N_1 S(\tau) + N_2[C(\tau) + \tau K S(\tau)], \tag{76}$$

where N_1 and N_2 are arbitrary constants.

From the way in which $S(\tau)$ and $C(\tau)$ were derived, it is readily seen that the coefficients of $\sin(j+1)\tau$ and $\cos(j+1)\tau$ in the coefficients of μ^j in $S(\tau)$ and $C(\tau)$, respectively, are equal.

§ 10. *Construction of the Solution for the Vertical Motion.*

Let

$$\xi = \sum_{j=1}^{\infty} \xi_j \epsilon^j, \tag{77}$$

where the ξ_j are to be periodic with the period 2π in τ.

Since the initial time was chosen for $\epsilon = 0$ so that $\dot{z}(0) = 0$, it may now be chosen so that $\dot{z}(0) = 0$ when $\epsilon \neq 0$. Now $z = \bar{z} + \xi$, and as $\dot{\bar{z}}(0) = 0$, it follows that $\dot{\xi}(0) = 0$. Hence

$$\dot{\xi}_j(0) = 0, \quad j = 1, \cdots \infty. \tag{78}$$

Let (77) be substituted in (67) and let the resulting equation be denoted by (67′). It is an identity in ϵ, and on equating the coefficients of the same powers of ϵ we obtain sets of differential equations which can be integrated and the various constants of integration can be chosen, as we shall show, so as to satisfy the periodicity and initial conditions.

Two types of series occur in this integration and they are similar to $S(\tau)$ and $C(\tau)$ in (71) and (73), respectively. These series will be denoted by

$$S_j(\tau),\ S^{(j)}(\tau),\ \bar{S}_j(\tau),\ \bar{S}^{(j)}(\tau),\ j = 1, 2, \cdots,$$

or

$$C_j(\tau),\ C^{(j)}(\tau),\ \bar{C}_j(\tau),\ \bar{C}^{(j)}(\tau),\ j = 1, 2, \cdots,$$

according as they are similar to $S(\tau)$ or $C(\tau)$, respectively.

Coefficients of ϵ. When the coefficients of ϵ to the first power are equated in (67′) we obtain

$$\ddot{\xi}_1 + \Theta_1 \dot{\xi}_1 + \Theta_2 \xi_1 = C^{(1)}(\tau). \tag{79}$$

The complementary function of this equation is the same as the solution of the equation of variation, viz.,

$$\xi_1 = n_1^{(1)} S(\tau) + n_2^{(1)} [C(\tau) + \tau K S(\tau)], \tag{80}$$

where $n_1^{(1)}$ and $n_2^{(1)}$ are the arbitrary constants. By employing the method of the variation of parameters to find the complete solution, we have

$$\left.\begin{aligned} &\dot{n}_1^{(1)} S(\tau) + \dot{n}_2^{(1)} [C(\tau) + \tau K\, S(\tau)] = 0, \\ &\dot{n}_1^{(1)} \dot{S}(\tau) + \dot{n}_2^{(1)} [\dot{C}(\tau) + K\{\tau \dot{S}(\tau) + S(\tau)\}] = C^1(\tau). \end{aligned}\right\} \tag{81}$$

The determinant of the coefficients of $\dot{n}_1^{(1)}$ and $\dot{n}_2^{(1)}$ in the above equations is the same as (75), and since it is different from zero equations (79) can be solved for $\dot{n}_1^{(1)}$ and $\dot{n}_2^{(1)}$. Thus

$$\left.\begin{aligned}\dot{n}_1^{(1)} &= -(1/\Delta)C^{(1)}(\tau)[C(\tau)+\tau K\,S(\tau)],\\ \dot{n}_2^{(1)} &= (1/\Delta)C^{(1)}(\tau)\,S(\tau).\end{aligned}\right\} \tag{82}$$

Since the coefficients of the same power of μ in $C^{(1)}(\tau)$ and $C(\tau)$ are sums of cosines of the same multiples of τ, the product $C^{(1)}(\tau)C(\tau)$ will yield, in addition to periodic terms, a constant

$$\mu p_1 = \mu[p_1^{(0)} + p_1^{(1)}\mu + \cdots + p_1^{(j)}\mu^j + \cdots\ ],$$

where $p_1^{(j)}$ are real constants. Then the integration of (82) gives

$$\left.\begin{aligned}n_1^{(1)} &= N_1^{(1)} - [\mu\tau p_1 + S^{(1)}(\tau) + \tau K\,\bar{C}^{(1)}(\tau)],\\ n_2^{(1)} &= N_2^{(1)} + \bar{C}^{(1)}(\tau),\end{aligned}\right\} \tag{83}$$

where $N_1^{(1)}$ and $N_2^{(1)}$ are the constants of integration. Since the coefficients of the sines and cosines of the highest multiples of τ in the coefficients of μ^j in $S(\tau)$ and $C(\tau)$, respectively, are equal, then it follows that $S^{(1)}(\tau)$ and $\bar{C}^{(1)}(\tau)$ have the same property. When (83) is substituted in (80) the complete solution of (79) then becomes

$$\xi_1 = N_1^{(1)}S(\tau) + N_2^{(1)}[C(\tau) + \tau KS(\tau)] - \mu\tau p_1 S(\tau) + (1/\mu)\bar{C}_1(\tau),$$

where $\bar{C}_1(\tau)$ contains no terms independent of μ. In order that ξ_1 shall be periodic $N_2^{(1)}$ must be given the value

$$N_2^{(1)} = \frac{\mu p_1}{K} = \frac{1}{\mu}\text{ (power series in } \mu), \tag{84}$$

and from the initial conditions (78) it follows that

$$N_1^{(1)} = 0.$$

Hence

$$\xi_1 = (1/\mu)C_1(\tau), \tag{85}$$

where $C_1(\tau)$, like $\bar{C}_1(\tau)$, contains no terms independent of μ.

Coefficients of ϵ^2. Equating the coefficients of ϵ^2 in (67) gives the differential equation

$$\ddot{\xi}_2 + \Theta_1\dot{\xi}_2 + \Theta_2\xi_2 = (1/\mu)C^{(2)}(\tau). \tag{86}$$

Except for the factor $1/\mu$ in the right member, this equation is similar to (79). The general solution is obtained in the same way as at the preceding step and is found to be

$$\xi_2 = N_1^{(2)} S(\tau) + N_2^{(2)} [C(\tau) + \tau K S(\tau)] + \tau p_2 S(\tau) + (1/\mu) \bar{C}^{(2)}(\tau),$$

where $N_1^{(2)}$ and $N_2^{(2)}$ are the constants of integration, and p_2 is a power series in μ. In order to satisfy the periodicity and initial conditions we must put

$$N_2^{(2)} = \frac{p_2}{K} = \frac{1}{\mu^2} \text{ (power series in } \mu), \quad N_1^{(2)} = 0.$$

Hence

$$\xi_2 = (1/\mu^2) C_2(\tau),$$

where $C_2(\tau)$, contains no terms independent of μ.

The remaining steps of the integration can be carried on in the same way and by an induction to the general term it can be shown that

$$\xi_n = (1/\mu^n) C_n(\tau),$$

where $C_n(\tau)$ as at the previous steps, contains no terms independent of μ.

On substituting the solutions for the various ξ_j in (77) we obtain

$$\xi = \sum_{j=1}^{\infty} C_j(\tau) (\epsilon/\mu)^j. \tag{87}$$

By applying Macmillan's theorem, quoted in § 3, it is found that the solution (87) converges for $|\epsilon|$ sufficiently small.

When (87) is substituted in (66), the solution for the vertical motion becomes

$$z = \bar{z} + \sum_{j=1}^{\infty} C_j(\tau) (\epsilon/\mu)^j. \tag{88}$$

Since ϵ and μ are both arbitrary we may put $\epsilon = \rho\mu$ and therefore (88) becomes

$$z = \bar{z} + \sum_{j=1}^{\infty} C_j(\tau) \rho^j. \tag{89}$$

This solution converges for $|\mu|$ and $|\rho|$ sufficiently small.

§ 10. *The Horizontal Motion.* When the first two equations of (3) are transformed by the substitution

$$t - t_0 = \tfrac{1}{2}\sqrt{(k/g)(1+\delta)}\ \tau, \quad \delta = -\tfrac{1}{2}\mu^2 + \cdots,$$

already used in (A), and the value of z obtained in (89) is substituted in 2λ, these differential equations become

$$\left.\begin{aligned} \ddot{x} + [\phi_0 + \phi_1\rho + \phi_2\rho^2 + \cdots]\, x = 0, \\ \ddot{y} + [\phi_0 + \phi_1\rho + \phi_2\rho^2 + \cdots]\, y = 0, \end{aligned}\right\} \tag{90}$$

where each ϕ_j is a power series in μ with sums of cosines of multiples of τ in the coefficients. The function ϕ_0 has the same value as the coefficient of x in (22).

Now let $\rho = \sigma\mu$. Then the coefficients in (90) can be rearranged as power series in μ and the differential equations take the same form as (22). Hence the solutions are

$$x = L_1\, e^{i\beta\tau}v_1 + L_2\, e^{-i\beta\tau}v_2,\ \ y = M_1\, e^{i\beta\tau}v_1 + M_2\, e^{-i\beta\tau}v_2, \tag{91}$$

where L_1, L_2, M_1, and M_2 are the constants of integration, and β, v_1, and v_2 are similar in form to a, u_1, and u_2, respectively, of (39), (52), or (60) according to the value of $\sqrt{k/p}$.

When the initial values $\dot{x}(0) = y(0) = 0$ are imposed on (91), it follows that

$$[i\beta + \dot{v}_1(0)]\ [L_1 - L_2] = 0,\ \ M_1 + M_2 = 0.$$

Since $i\beta + \dot{v}_1(0) \neq 0$ for $|\mu|$ sufficiently small, then

$$L_1 = L_2 = L, \text{ say},$$

and, from the second equation,

$$M_1 = -M_2 = M, \text{ say}.$$

Hence the solutions (91) become

$$x = L[e^{i\beta\tau}v_1 + e^{-i\beta\tau}v_2],\ \ y = M[e^{i\beta\tau}v_1 - e^{-i\beta\tau}v_2].$$

The constants L and M can be determined as in § 7, and it is found that L is real while M is purely imaginary. Suppose $M = iN$. Then

$$x = 2L[\cos\beta\tau\{1 + \sum_{j=1}^{\infty}\sum_{l=0}^{j}\mu^j a_l^{(j)}\cos l\tau\} - \sin\beta\tau\{\sum_{j=1}^{\infty}\sum_{l=1}^{j}\mu^j b_l^{(j)}\sin l\tau\}],$$
$$y = 2N[\sin\beta\tau\{1 + \sum_{j=1}^{\infty}\sum_{l=0}^{j}\mu^j a_l^{(j)}\cos l\tau\} - \sin\beta\tau\{\sum_{j=1}^{\infty}\sum_{l=1}^{j}\mu^j b_l^{(j)}\sin l\tau\}],$$

where $a_l^{(j)}$ and $b_l^{(j)}$ are real constants. These equations together with equation (89) are the solutions of the equations of motion of the problem under consideration.

QUEEN'S UNIVERSITY,
KINGSTON, CANADA.

On the Convergence of Certain Classes of Series of Functions.

BY R. D. CARMICHAEL.

§ 1. INTRODUCTION.

Let $v_n(x)$, $n=0,1,2,\cdots$, be an infinite sequence of functions of x which may be written in the form

$$v_n(x)=\sum_{i=0}^{\mu_1}\sum_{j=0}^{\nu_1} a^{(n)}_{\mu-i,\,\nu-j}\quad x^{\mu-i}(\log x)^{\nu-j}+\frac{M_n(x)}{x^{\mu_1-\mu+1}(\log x)^{\nu_1-\nu+1}} \tag{1}$$

where μ, μ_1, ν, ν_1 are integers such that $\mu_1 \geqq \mu$, $\nu_1 \geqq \nu$, and where $M_n(x)$, for fixed n, is a function of x which in absolute value is not greater than a constant M_n when $|x|$ is greater than some constant X and x is in a sector V formed by two rays proceeding from zero to infinity and including the positive axis of reals between them. Let $M_n(x)$, for every n, be an analytic function of x in every finite portion of the region of the x-plane just defined.

Moreover, let one of the coefficients a in (1), say $a^{(n)}_{kl}$ where k and l are not simultaneously zero, possess the following properties:

1) As n becomes infinite $a^{(n)}_{kl}$ becomes infinite, while its argument approaches a finite limit, say that it comes to coincide with the argument of a given constant σ, so that it may be written $a^{(n)}_{kl}=\sigma(\alpha^{(n)}+i\beta^{(n)})$ where $\alpha^{(n)}$ and $\beta^{(n)}$ are real. Suppose, moreover, that $\alpha^{(n)}$ is monotonic increasing when n is greater than some given constant. (It is easy to see that $\beta^{(n)}/\alpha^{(n)}$ approaches zero as n becomes infinite, a fact for which we shall have use later.)

2) The coefficient $a^{(n)}_{kl}$ has a dominance property of such sort that

$$\lim_{n=\infty} M_n(x)/a^{(n)}_{kl}=0$$

for every x for which $M_n(x)$ is analytic and

$$\lim_{n=\infty} a^{(n)}_{ij}/a^{(n)}_{kl}=0$$

unless simultaneously $i=k$ and $j=l$.

3) When $v_n(x)$ is one-termed so that it has the special value $v_n(x)=a^{(n)}_{kl}x^k(\log x)^l$ and $a^{(n)}_{kl}$ is real we make no further hypothesis; otherwise we

suppose that a positive constant ϵ_1 exists such that when n is greater than some appropriate N_{ϵ_1} we have $a^{(n+1)} - a^{(n)} \geqq \epsilon_1$.

Then let us consider the series of functions

$$S(x) = \sum_{n=0}^{\infty} c_n e^{v_n(x)} \tag{2}$$

where $c_0, c_1, c_2 \cdots$ are constants.

A point $\bar{x}$ will be called an exceptional or a non-exceptional point for the series $S(x)$ according as $\bar{x}$ is or is not a singularity of $v_n(x)$ for some value of n.

The principal object of the present paper is to consider the central convergence problem for the series $S(x)$ and for a certain other series defined in § 4 and having similar properties. It turns out that the convergence theory of these series may be readily developed in an elementary way. The character of the region of convergence and the uniform convergence of series $S(x)$ are treated in § 2. In § 3 theorems are established showing the coincidence in special cases of the regions of convergence of series $S(x)$ depending on different sequences $v_n(x)$. In § 4 are developed corresponding properties of a second class of series there defined. In § 5 cases are considered in which series of either class define functions having formal power series expansions.

A considerable variety of important classes of series are included in the general form (2). We shall exhibit a few of these.

(a) Let $v_n(x) = n \log x$. Our series is then an ascending power series. If we take $v_n(x) = -n \log x$ we have the descending power series.

(b) If we put $v_n(x) = -\lambda_n x$ where $\lambda_0, \lambda_1, \lambda_2, \cdots$ is an increasing sequence of real numbers tending to infinity then $S(x)$ is the generalized Dirichlet series. The extension of Dirichlet series employed in *Transactions American Mathematical Society* 17 (1916), p. 218, is also essentially included here.

(c) If we take $v_n(x) = \log \Gamma(x) - \log \Gamma(x+n)$, so that our series becomes the factorial series, it may be shown that $v_n(x)$ fulfills the requisite conditions, the asymptotic formula for $\Gamma(x)$ serving readily for this purpose.

(d) Indeed the more general class of series $\Sigma\, c_n g(x+n)$ which I have treated in *Trans. Amer. Math. Soc.* 17 (1916), pp. 207-232, may be shown to belong to the class treated here.

In the convergence proofs we shall have need of two lemmas which are reproduced here for the reader's convenience (for reference to the proofs of these lemmas see page 211 of the paper just cited):

LEMMA I. *Let $a_0 + a_1 + a_2 + \cdots$ be a convergent series of constants and let $\beta_0, \beta_1, \beta_2, \cdots$ be an infinite sequence of numbers such that the series*

$\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ *is convergent. Then the series* $a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + \cdots$ *is convergent.*

Lemma II. *Let* $a_0 + a_1 + a_2 + \cdots$ *be a convergent series of constants and let* $\beta_0, \beta_1, \beta_2, \cdots$ *be an infinite sequence of functions of the complex variable* x *analytic in a given closed domain* D *and such that the series* $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ *converges uniformly in* D. *Then the series* $a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + \cdots$ *converges uniformly in* D.

§ 2. *Character of the region of convergence of* $S(x)$.

Let us suppose that the series $S(x)$ converges for a given non-exceptional value x_0 of x, say briefly that $S(x_0)$ converges; and let us seek conditions on the non-exceptional value x_1 of x sufficient to ensure that $S(x_1)$ shall be convergent. We employ lemma I, taking

$$a_n = c_n\, e^{v_n(x_0)}, \quad \beta_n = e^{v_n(x_1) - v_n(x_0)}.$$

The series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, whose convergence is sufficient to ensure the convergence of $S(x_1)$, may be put in the form

$$\sum_{n=N}^{\infty} \left| e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \right| r_n \tag{3}$$

where

$$r_n = \frac{\left| e^{v_{n+1}(x_1) - v_{n+1}(x_0)} - e^{v_n(x_1) - v_n(x_0)} \right|}{\left| e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \right|},$$

τ having the value

$$\tau = \theta\, \sigma\, \{x_1^k(\log x_1)^l - x_0^k(\log x_0)^l\},$$

where θ is a positive constant not greater than unity. We shall now suppose that the real part $R(\tau)$ of τ is negative.

We shall now show that a proper choice of θ will bring it about that r_n is bounded. If $v_n(x)$ is one-termed in the sense of condition 3) then we take $\theta = 1$, whence $r_n = 1$; otherwise we take θ to be less than unity. Then if we divide the numerator and the denominator of the fraction r_n by $\left| e^{a^{(n)}\tau} \right|$ we have left in the denominator a quantity bounded away from zero for every n in (3) is N is taken sufficiently large, as one sees from the condition in hypothesis 3). Moreover, each of the two exponential terms in the numerator of this fraction, and hence this numerator itself, approaches zero as n becomes infinite, as one sees readily through use of 1), 2) and 3) and

particularly of the dominance property of $a_{kl}^{(n)}$. Hence in any case r_n is bounded.*

It follows therefore that series (3) converges provided that the series

$$\sum_{n=N}^{\infty} \left| e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \right|$$

is convergent.

To prove the convergence of this series we observe that

$$e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} = (1/\tau) \int_{a^{(n)}}^{a^{(n+1)}} e^{u\tau} du$$

so that

$$\left| e^{a^{(n+1)}\tau} - e^{a^{(n)}\tau} \right| \leqq (1/|\tau|) \int_{a^{(n)}}^{a^{(n+1)}} e^{uR(\tau)} du.$$

The series of which the nth term is the second member of this relation is obviously convergent since $a^{(n)}$ is ultimately monotonic and $R(\tau) < 0$ on account of the condition imposed on x_1.

Hence we have the part of the following theorem which refers to convergence (not absolute convergence):

Theorem I. *Let x_0 and x_1 be two values of x which are non-exceptional for the series $S(x)$ and suppose that $S(x_0)$ converges [converges absolutely]: Then $S(x_1)$ also converges [converges absolutely] provided that*

$$R\{\sigma x_1^{k} (\log x_1)^{l}\} < R\{\sigma x_0^{k} (\log x_0)^{l}\}.$$

The proof of the part of the theorem referring to absolute convergence is immediate. In fact it is sufficient to show that the ratio ${}_e v_n(x_1)/{}_e v_n(x_0)$ is bounded as n becomes infinite; and this is an immediate consequence of the hypotheses on $v_n(x)$.

By a *region C of convergence* of the series $S(x)$ we shall mean a region such that $S(x)$ converges for every non-exceptional value of x in the interior of C and diverges for every non-exceptional value of x exterior to C. In a similar way we define a *region Γ of absolute convergence of $S(x)$*.

By means of theorem I and the application of a classic method it is easy to determine the character of the regions of convergence and absolute convergence of $S(x)$. Compare the similar argument on p. 214 of the memoir already cited. We have the following result:

* It should be observed that the only use of condition 3) in this proof is that made in showing that r_n is bounded, so that the theorem obtained is true when r_n is bounded even though condition 3) should not be satisfied.

Theorem II. *There exists a unique real number* λ [μ] *such that the region of convergence* [*absolute convergence*] *of the series* $S(x)$ *is bounded by the curve*

$$R\{\sigma x^k (\log x)^l\} = \lambda \quad [= \mu] \tag{4}$$

and lies on that side of this curve for which $R\{\sigma x^k (\log x)^l\}$ *is less than* λ [μ].

By the use of lemma II and a modification, mostly verbal in character, of the argument by which theorem I was established we may prove the following theorem:

Theorem III. *The series* $S(x)$ *converges uniformly in any closed domain* D *which lies within its region of convergence and contains no point which is exceptional for* $S(x)$ *or is a limit point of points which are exceptional for* $S(x)$.

As an immediate consequence of this we have the following:

Theorem IV. *The sum of the series* $S(x)$ *is a function* $S(x)$ *of* x *which is analytic at every non-exceptional point which is in the interior of its region of convergence and is not a limit point of exceptional points; and the derivatives of* $S(x)$ *at every such point may be found by differentiating the series* $S(x)$ *term by term.*

We should examine briefly the nature of the curves defined by equations of the form (4). For the case when $l = 0$ and k is positive I have already briefly described them in *Bull. Amer. Math. Soc.* (2) 23 (1917), pp. 424–425. In this case they are obviously algebraic. In particular, when $l = 0$ and $k = 1$ the curve is a straight line.

Suppose next that $l = 0$ and k is a negative integer $-t$. Then the curves have equations of the form $R(\sigma x^{-t}) = \eta_1$ and are again algebraic. If we write $\sigma = |\sigma| e^{i\phi}$, $\eta = \eta_1/|\sigma|$ and $x = re^{i\theta}$ where r is real and not negative, the equation of our curve may be written in polar coördinates r and θ in the form

$$\eta r^t = \cos(\phi - t\,\theta).$$

In case $\eta = 0$ our curve consists of $2t$ rays proceeding from 0 to ∞ and dividing the angular space about zero into $2t$ equal parts or sectors. The quantity $\cos(\phi - t\,\theta)$ is negative within alternate sectors of this set (the sectors of convergence) and positive within the others (in general the sectors of divergence). When η is not zero the curve consists of t branches lying within alternate sectors of the preceding set of $2t$ sectors, and in those for which $\cos(\phi - t\,\theta)$ is negative or positive according as η is negative or positive. For the case when $t = 1$ this curve is a circle.

When $k=0$, $l=1$ and σ is real equation (4) may be written in the form $R[\log x]=\eta_2$ or $|x|=\eta$, so that the curve is a circle about 0 as a center. As we have already seen, this includes the case when $S(x)$ is a power series either ascending or descending.

Except when $l=1$ and $k=0$, or $l=0$ and k is unrestricted, the curves (4) are in general transcendental.

§ 3. *Coincidence of the regions of convergence of different series $S(x)$.*

In connection with series $S(x)$ let us consider the two related series

$$S_{kl}(x)=\sum_{n=0}^{\infty} c_n e^{a_{kl}^{(n)}x^k(\log x)^l}; \qquad \bar{S}_{kl}(x)=\sum_{n=0}^{\infty} c_n e^{\sigma a^{(n)}x^k(\log x)^l}.$$

It is clear that each of these is a series of the general class $S(x)$ defined in § 1. Consequently the four theorems already established are valid for these series also. In the present section we establish relations among them and the more general series $S(x)$.

Theorem V. *The boundary curve of the region of convergence [absolute convergence] is the same for the three series $S(x)$, $S_{kl}(x)$, $\bar{S}_{kl}(x)$.*

It should be observed that the theorem says nothing about convergence [absolute convergence] on the boundary of the region of convergence [absolute convergence].

It is obviously sufficient to prove the theorem for the case of any two pairs of the three series, say for $S(x)$ and $\bar{S}_{kl}(x)$, and for $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. In fact, it is enough to prove it for the first of these pairs. The method of proof is identical in the two cases, so that it is sufficient to carry out the work for either one of them alone. For the latter pair the formulae are somewhat simpler than for the former, so that we shall give in detail the proof of only that part of the theorem which relates to the series $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. This proof falls into two parts.

1. Let $\bar{x}$ be any non-exceptional point in the interior of the region of convergence of $S_{kl}(x)$. We shall prove that $\bar{x}$ is likewise in the interior of the region of convergence of $\bar{S}_{kl}(x)$.

If $R\{\sigma x^k(\log x)^l\}=\lambda$ is the boundary of the region of convergence of $S_{kl}(x)$, then $R\{\sigma\bar{x}^k(\log\bar{x})^l\}<\lambda$. Then it is obvious that non-exceptional numbers x_0 and x_1 exist such that

$$R\{\sigma\bar{x}^k(\log\bar{x})^l\}<R\{\sigma x_1^k(\log x_1)^l\}<R\{\sigma x_0^k(\log x_0)^l\}<\lambda. \qquad (5)$$

Then x_0 and x_1 are points of convergence of $S_{kl}(x)$. Taking

$$\alpha_n = c_n e^{a_{kl}^{(n)} x_0^k (\log x_0)^l}, \qquad \beta_n = e^{\sigma a^{(n)} x_1^k (\log x_1)^l - a_{kl}^{(n)} x_0^k (\log x_0)^l};$$

and applying lemma I we see that $\bar{S}_{kl}(x_1)$ converges if $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges. Now the exponent in the value of β_n may be written in the form

$$a^{(n)}\{\sigma x_1^k (\log x_1)^l - \sigma x_0^k (\log x_0)^l\} - \beta^{(n)} i \sigma x_0^k (\log x_0)^l.$$

We can now proceed by the method employed for a like matter near the beginning of § 2 and show that $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges. Hence $\bar{S}_{kl}(x_1)$ converges; and therefore $\bar{S}_{kl}(\bar{x})$ converges, as one sees through theorem I and the relation between $\bar{x}$ and x_1 in (5); and $\bar{x}$ is in fact in the interior of the region of convergence of $\bar{S}_{kl}(x)$.

2. If we suppose next that $\bar{x}$ is in the interior of the region of convergence of $\bar{S}_{kl}(x)$ it may be shown that it is likewise in the interior of the region of convergence of $S_{kl}(x)$. For this it is sufficient to apply lemma I as in the preceding case, taking this time for α_n and β_n the values

$$\alpha_n = c_n e^{\sigma a^{(n)} x_0^k (\log x_0)^l}, \qquad \beta_n = e^{a_{kl}^{(n)} x_1^k (\log x_1)^l - \sigma a^{(n)} x_0^k (\log x_0)^l},$$

where x_0 and x_1 again satisfy relations (5), the curve $R\{\sigma x^k (\log x)^l\} = \lambda$ being now the boundary of the region of convergence of $\bar{S}_{kl}(x)$.

From the conclusions of the two preceding paragraphs we see that theorem V is true in so far as it relates to the region of convergence of $S_{kl}(x)$ and $\bar{S}_{kl}(x)$. The part relating to the region of absolute convergence of the same two series may be proved in the same way; or it may be proved more directly and more easily by a term-by-term comparison of the absolutely convergent series $S_{kl}(x_0)$ and $\bar{S}_{kl}(x_1)$ in the first case and $S_{kl}(x_1)$ and $\bar{S}_{kl}(x_0)$ in the second case, x_0 and x_1 being connected with an interior point $\bar{x}$ of the region of absolute convergence by a relation of the form (5), the curve $R\{\sigma x^k (\log x)^l\} = \lambda$ being now the boundary of the region of absolute convergence.

Theorem V thus established brings out the fact that for a given set of coefficients $c_0, c_1, c_2, \cdots$ in $S(x)$ the functions $v_n(x)$, subject to the permanent hypotheses as to character, can be modified in any way whatever so long as $a_{kl}^{(n)}$ is left unchanged (and indeed so long as σ and $a^{(n)}$ are left unchanged) without disturbing the position of the boundary curve of the region of convergence. Such changes may introduce or remove exceptional points or modify their character; but beyond this it affects convergence [absolute convergence] only on the boundary of the region.

Again, *Theorem V affords us a satisfying means of finding the convergence number λ [the absolute convergence number μ] of the series $S(x)$.* In

fact, it is the negative of the convergence abscissa [absolute convergence abscissa] of the Dirichlet series

$$D(t)=\sum_{n=0}^{\infty} c_n e^{-a^{(n)}t}$$

as one sees by comparison with $\bar{S}_{kl}(x)$ where $-t$ is thought of as replacing $\sigma x^k(\log x)^l$. Convenient formulae for the convergence abscissa of Dirichlet series are quoted or otherwise referred to on pages 224-225 of my memoir already cited.

§4. *Similar properties of a second class of series.*

Let us now consider similar questions for a series of the form

$$T(x)=c_0+\sum_{n=1}^{\infty} c_n P_1(x)P_2(x)\cdots P_n(x),$$

where $c_0, c_1, c_2, \cdots$ are constants and $P_1(x), P_2(x), P_3(x), \cdots$ a given sequence of functions.

A point $\bar{x}$ will be called exceptional for the series $T(x)$ if any one of the functions $P_n(x)$ has a singularity or a zero at $\bar{x}$; otherwise it will be called non-exceptional.

Let x_0 be a non-exceptional point such that $T(x_0)$ converges and let x_1 be a second non-exceptional point. Consider what relation between x_1 and x_0 is sufficient to ensure the convergence of the series $T(x_1)$. We employ lemma I, taking,

$$a_n=c_nP_1(x_0)P_2(x_0)\cdots P_n(x_0),\quad \beta_n=\frac{P_1(x_1)\cdots P_1(x_1)}{P_1(x_0)\cdots P_n(x_0)}.$$

Then $T(x_1)$ converges if the series $\Sigma_n\,|\,\beta_{n+1}-\beta_n\,|$ converges. Now the ratio $R_n(x_1, x_0)$ of two consecutive terms of this series (the nth to the $(n-1)$th) may be put in the form

$$R_n(x_1,x_0)=\left|\frac{P_n(x_1)}{P_n(x_0)}\right|\cdot\frac{\left|\dfrac{P_{n+1}(x_1)}{P_{n+1}(x_0)}-1\right|}{\left|\dfrac{P_n(x_1)}{P_n(x_0)}-1\right|}.$$

Denote by $l(x_1, x_0)$ the greatest limit (the superior limit) of $R_n(x_1, x_0)$ as n becomes infinite.

Now if $l(x_1, x_0)$ depends explicitly upon x_1 and we determine x_1 so that $l(x_1, x_0)\leqq 1-\epsilon$, where ϵ is a positive quantity, we are assured that our series $\Sigma_n\,|\,\beta_{n+1}-\beta_n\,|$, and hence that our series $T(x_1)$, converges. Moreover, if this inequality holds uniformly for x_1 in a given closed region containing no

exceptional points and no limit point of exceptional points either in its interior or on its boundary, then the series $T(x_1)$ converges uniformly in this region. Moreover, if $l(x_1, x_0)$ can be written in the form $\theta(x_1)/\theta(x_0)$, then it is easy to show that a number λ exists such that $\theta(x) = \lambda$ is the boundary of the region of convergence in the sense that $T(x)$ converges for every non-exceptional point for which $\theta(x) < \lambda$ and diverges for every non-exceptional point for which $\theta(x) > \lambda$. Again if $l(x_1, x_0)$ can be written in the form $\theta(x_1) - \theta(x_0) + 1$, we may likewise readily derive a similar result.

Now if $l(x_1, x_0)$ is independent of x_1 and x_0 and is greater than or equal to unity we get no information concerning the convergence of $T(x_1)$. But if $l(x_1, x_0)$ is independent of x_1 and x_0 and has a value less than unity we conclude that $T(x_1)$ is convergent without further restriction on x_1. Series of sort therefore have the interesting property that if they converge for a single non-exceptional value of x they converge for every non-exceptional value.

In one of the cases in which the foregoing argument fails, namely, that in which $\lim_{n=\infty} R_n(x_1, x_0) = 1$, we may profitably proceed to a consideration of the greatest limit of the quantity

$$n\{R_n(x_1, x_0) - 1\}.$$

We denote this greatest limit by $l_1(x_1, x_0)$. If it is less than or equal to $-1 - \epsilon$ where ϵ is a positive constant, then the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ converges, whence we conclude that $T(x_1)$ also converges.

Now if $l_1(x_1, x_0)$ depends on x_1 and we choose x_1 so that $l_1(x_1, x_0) \leqq -1 - \epsilon$ we have a situation similar to that just treated above and as before we can proceed readily to the determination of the character of the region of convergence, at least when $l_1(x_1, x_0)$ can be written as the quotient of a function of x_1, by a function of x_0, or $l_1(x_1, x_0) - 1$ as a difference of such functions.

Again, when $l_1(x_1, x_0)$ is independent of x_1 and x_0 we may treat the problem in the way indicated for the similar case above.

Out of other general criteria for the absolute convergence of series, as applied to the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, we may derive other related results. Those which we have stated have actually arisen frequently in special form in the investigation of the convergence of particular classes of series.*

It is desirable to examine certain special cases in which the foregoing

* One desiring to examine these special cases will find some of them treated and the others referred to in two papers of mine, namely, those in *Bull. Amer. Math. Soc.* (2) 8 (1917): 407-425 and AMER. JOURN. MATH., 36 (1914): 267-288, and in a paper by E. Cotton in *Bull. Soc. Math. Fr.*, 46 (1919): 69-84. The last paper is interesting for its general theorems, some of which are to be associated with the results of this section.

greatest limits exist in such way as to give rise to an elegant theory.

Let us suppose that

$$\lim_{n=\infty} \frac{P_n(x_1)}{P_n(x_0)}$$

exists, and let us denote its value by $m(x_1, x_0)$. Then if $m(x_1, x_0) \neq 1$ we have $\lim_{n=\infty} R_n(x_1, x_0) = | m(x_1, x_0) |$ so that $| m(x_1, x_0) |$ is to be identified with the $l(x_1, x_0)$ of the preceding discussion. An instance of this sort is afforded by a certain class of expansions in polynomials. Thus if we have

$$P_n(x) = a_{1n}(x-a) + a_{2n}(x-a)^2 + \cdots + a_{kn}(x-a)^k + \cdots a_{mn}(x-a)^m$$

where one of the coefficients a_{kn} dominates the others (in case there are any) in the sense that the quotient of any other by a_{kn} approaches zero as n becomes infinite, it is clear that $m(x_1, x_0) = (x_1 - a)^k/(x_0 - a)^k$. We see readily that the region of convergence is bounded by a circle about the point a as a center. For the special case of this in which $P_n(x) = x - a$ we have the usual ascending power series in $x - a$. By taking $P_n(x)$ a polynomial in $(x-a)^{-1}$ we obtain a like generalization of the usual descending power series in $x - a$.

This obviously may be further generalized by taking $P_n(x)$ in the form

$$P_n(x) = a_{1n}u_1(x) + \cdots + a_{kn}u_k(x) + \cdots + a_{mn}u_n(x)$$

where $u_1(x), \cdots, u_m(x)$ are given functions of x and the coefficient a_{kn} dominates the others in the same sense as before. The boundary of the region of convergence is now defined by an equation of the form $| u_k(x) | = \lambda$, and the region of convergence is on that side of this curve for which $| u_k(x) | < \lambda$. [It is clear that the finite series for $P_n(x)$ may be replaced by an infinite series if suitable hypotheses are made as to the character of its convergence.]

Let us now consider the case in which the foregoing limit value $m(x_1, x_0)$ is unity. Suppose that $P_n(x_1)/P_n(x_0)$ may be written in the form

$$\frac{P_n(x_1)}{P_n(x_0)} = 1 + \frac{m_1(x_1, x_0)}{n} + \frac{\xi_n(x_1, x_0)}{n^{1+\epsilon}},$$

where ϵ is a positive constant and $\xi_n(x_1, x_0)$ is bounded when n becomes infinite. Then we have

$$\begin{aligned}\lim_{n=\infty} n\{R(x_1, x_0) - 1\} &= \lim_{n=\infty} n \left[\left| \frac{P_n(x_1)}{P_n(x_0)} \right| \cdot \left| \frac{\dfrac{m_1(x_1, x_0)}{n+1} + \dfrac{\xi_{n+1}(x_1, x_0)}{(n+1)^{1+\epsilon}}}{\dfrac{m_1(x_1, x_0)}{n} + \dfrac{\xi_{n+1}(x_1, x_0)}{n^{1+\epsilon}}} \right| - 1 \right] \\ &= \lim_{n=\infty} n \left[\left\{ 1 + \frac{R\{m_1(x_1, x_0)\}}{n} \right\} \{1 - (1/n)\} - 1 \right] \\ &= R\{m_1(x_1, x_0)\} - 1.\end{aligned}$$

Hence $R\{m_1(x_1, x_0)\} - 1$ may be identified with the limit $l_1(x_1, x_0)$ of the preceding more general treatment; and the relevant convergence properties are therefore expressible in simple and elegant form.

It may be observed that it is essentially the Gauss criterion of convergence which lies at the bottom of the special result just obtained. Significant extensions of this result may be secured, if needed, by using a more delicate criterion than that of Gauss, say one of the infinite sequence of criteria due to Kummer (see *Encyclopédie des Sc. Math.*, I, p. 223).

The condition last indicated in detail is realized in the case of factorial series. Here we have $P_n(x) = 1/(x+n-1) = \Gamma(x+n-1)/\Gamma(x+n)$ so that

$$\frac{P_n(x_1)}{P_n(x_0)} = \frac{x_0+n-1}{x_1+n-1} = 1 + \frac{x_0 - x_1}{n} + \frac{\xi_n(x_1, x_0)}{n^2}.$$

Hence if the factorial series converges for a non-exceptional point x_0 it converges also for the non-exceptional point x_1 if $R(x_1) > R(x_0)$, as is well known.

It may be seen also that this condition is simply realized in a much more general class of cases. Let us put

$$P_n(x) = \frac{g(x+n)}{g(x+n-1)}$$

where $g(x)$ is a given function possessing the asymptotic expansion

$$g(x) \sim x^{\mu+\sigma x} e^{\alpha+\beta x}(1 + (a_1/x) + \cdots), \qquad \sigma \neq 0,$$

valid in a sector V formed by two rays proceeding from 0 to ∞ and including between them the positive axis of reals. Then we have as to n an asymptotic relation of the form

$$P_n(x) \sim n^{\sigma} e^{\sigma-\beta}(1 + (\mu + \sigma x)/n + \cdots);$$

whence

$$\frac{P_n(x_1)}{P_n(x_0)} \sim 1 + \frac{\sigma(x_1 - x_0)}{n} + \cdots$$

From this we conclude that our series $T(x_1)$ in this case converges if $R(\sigma x_1) < R(\sigma x_0)$, and that the boundary of its region of convergence is a straight line $R(\sigma x) = \lambda$.

There is another range of cases, of which the generalized Dirichlet series affords an example, in which one may readily conclude to the convergence of

$\Sigma_n \mid \beta_{n+1} - \beta_n \mid$ and hence of $T(x_1)$. Let us suppose that $\beta_{n+1} - \beta_n$ may be written in the form of an integral

$$\beta_{n+1} - \beta_n = \int_{C_n} u(t, x_1, x_0)\, dt$$

where C_n is a finite path of integration for each n, no two of these paths having a common arc. Let C be any path made up of all the paths C_l, C_{l+1} C_{l+2}, $\cdots$ (where l is a given integer) and any other paths which it is convenient to include. Then if the integral

$$\int_C \mid u(t, x_1, x_0) \mid dt$$

exists when x_1 is related in a specified way to x_0, it is clear that the series $\Sigma_n \mid \beta_{n+1} - \beta_n \mid$, and hence the series $T(x_1)$, converges under the same hypothesis as to the relation of x_1 and x_0.

If we take

$$P_1(x) = e^{-\lambda_1 x}, \quad P_n(x) = e^{-\lambda_n x + \lambda_{n-1} x} \text{ when } n > 1,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ is a monotone increasing sequence such that λ_n becomes infinite with n, our series $T(x)$ with $c_0 = 0$ is the Dirichlet series and the method indicated may be applied in the classic way illustrated already in § 2.

In the problem of representing functions with given properties or of finding functions with anticipated properties in given regions, it is sometimes desirable to have representations of them valid in certain preassigned regions of the plane. It is therefore of interest to ask under what simple conditions one will have a region of convergence of specified form: We add here a few remarks on this matter.

One of the simplest regions of convergence is a half plane bounded by a straight line. We have already observed certain cases in which the region of convergence is of this form, having the equation $R(\sigma x) = \lambda$. This suggests a more general case in which the same type of region of convergence arises. Let us suppose that the limit $m_1(x_1, x_0)$ of the preceding discussion is such that we have the relation

$$R\{m_1(x_1, x_0)\} = R\{\sigma(x_1 - x_0)\} \cdot \eta(x_1, x_0),$$

where $\eta(x_1, x_0)$ is positive whatever x_1 and x_0 are. Then it is easy to see that the region of convergence is bounded by a curve whose equation is of the form $R(\sigma x) = \lambda$.

From this special case it is clear that we may determine circumstances under which any one of a great variety of curves may be realized as the

boundary of the region of convergence and that we may attach this discussion to any one of the four limit quantities $l(x_1, x_0)$, $l_1(x_1, x_0)$, $m(x_1, x_0)$, $m_1(x_1, x_0)$ employed above. We have already seen in particular how circular regions may be realized in a great variety of instances. For other circular regions and half-plane regions, see my papers already cited and the papers referred to in them.

§ 5. *Cases in which the Series Define Functions Having Formal Power Series Expansions.*

Owing to the great importance, in the theory of differential and difference equations and in other parts of analysis, of functions possessing formal power series expansions either convergent or divergent, it is desirable to know the circumstances under which our series $T(x)$ and $S(x)$ can be formally transformed into power series and these conversely into series $T(x)$ or $S(x)$.

Let us consider the case of series $T(x)$ where $P_n(x)$ has the descending formal power series expansion

$$P_n(x) = \frac{c_{1n}}{x} + \frac{c_{2n}}{x^2} + \frac{c_{3n}}{x^3} + \cdots, \qquad n = 1, 2, 3, \cdots, \tag{7}$$

where C_{1n} is different from zero for every value of n. It is obvious that the series $T(x)$ is then transformable formally into a descending power series in x and that the coefficients may be reckoned out by means of readily solvable recurrence relations.

A simple case of such a series $T(x)$ is the factorial series in which $P_n(x) = 1/(x + n - 1)$. The generalization of factorial series introduced in § 4 by aid of the function $g(x)$ also belongs here when $\sigma = -1$, as one may show without difficulty. If we should take σ to be the negative reciprocal of an integer k we should have a generalization to the case in which the formal descending power series in x are replaced by formal descending power series in the kth root of x.

If we have two functions $T_1(x)$ and $T_2(x)$ defined by two series $T(x)$ each depending on the set of functions (7) and if the product $T_1(x)T_2(x)$ is expansible into a series $T(x)$ of the same form, then the coefficients of the expansion may be found in the following manner: Transform the series for $T_1(x)$ and $T_2(x)$ into formal descending power series in x, take the product of these latter and transform it formally into a series $T(x)$; this series will represent the product $T_1(x)T_2(x)$. This process will certainly be valid at least when $T_1(x)$ and $T_2(x)$ are asymptotic to their formal power series representations and the functions P are such that no function has two representa-

tions in the form of a series $T(x)$. An instance of this sort is afforded by the series defined in § 4 in terms of $g(x)$, provided that $\sigma = -1$, as one sees from the results in AMER. JOURN. MATH. 39 (1917): 385-403. It is not difficult to see that the case just treated is an instance of series $S(x)$ as well as of series $T(x)$.

UNIVERSITY OF ILLINOIS,
September, 1919.

On the Solution of Linear Equations in Infinitely Many Variables by Successive Approximations.*

By J. L. Walsh.

In this paper we shall consider systems of equations of the type

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = c_1\ , \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots = c_2\ , \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots = c_3\ , \\ \cdots\cdots\cdots\cdots\cdots\ , \end{array}\right\} \tag{1}$$

where the a_{ij} and c_i are given real or complex numbers and the x_i are to be determined. Systems of type (1) have been solved by various means,† including the method of successive approximations, but this method has been used chiefly for Hilbert space [i. e., the space of points $\{x_k\}$ for which $\sum_{k=1}^{\infty} |x_k|^2$ converges] with corresponding restrictions on the a_{ij} and c_i.‡ It is the object of the present paper to give a number of new conditions under which (1) can be solved by successive approximations; in particular, it is shown that if (1) has a non-vanishing normal determinant and if a simple transformation of the system is made, then the method of successive approximations can be used. The method of successive approximations is very convenient for numerical computation.

We shall use the method of successive approximations to prove the following theorem, which applies to a system of equations of type slightly less general than (1):

Theorem I. *If there exist positive constants C, M, and P such that the coefficients of the system*

$$\left.\begin{array}{r} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = c_1\ , \\ x_2 + a_{23}x_3 + \cdots = c_2\ , \\ x_3 + \cdots = c_3\ , \\ \cdots\cdots\ , \end{array}\right\} \tag{2}$$

satisfy the restrictions

* Presented to the American Mathematical Society (Chicago), Apr. 7, 1917.

† See, e. g., F. Riesz, *Équations Linéaires.*

‡ See E. Goldschmidt, Würzburg Dissertation (1912).

$|c_k| \leqq MC^k$, $\sum_{j=k+1}^{\infty} |a_{kj}|$ *convergent* $(k = 1, 2, \cdots)$, $\sum_{j=k+1}^{\infty} |a_{kj}| \leqq P$ *for every* $k > K$, $C < (1/P)$, $C \leqq 1$, *then* (2) *has one solution and only one solution for which* $|x_k| \leqq \mu\gamma^k$, $\gamma < (1/P)$, $\gamma \leqq 1$.

We first consider the special case $K = 0$, and we take for approximations

$$\left.\begin{aligned} &x_k^{(1)} = c_k \quad (k = 1, 2, \cdots), \\ &x_k^{(i+1)} = c_k - [a_{kk+1}x_{k+1}^{(i)} + a_{kk+2}x_{k+2}^{(i)} + \cdots] \quad (i = 1, 2, \cdots). \end{aligned}\right\} \tag{3}$$

From (3) it follows that

$$\left.\begin{aligned} x_k^{(1)} &= c_k, \\ x_k^{(2)} - x_k^{(1)} &= -[a_{k,k+1}x_{k+1}^{(1)} + a_{k,k+2}x_{k+2}^{(1)} + \cdots], \\ x_k^{(3)} - x_k^{(2)} &= -[a_{k,k+1}(x_{k+1}^{(2)} - x_{k+1}^{(1)}) + a_{k,k+2}(x_{k+2}^{(2)} - x_{k+2}^{(1)}) + \cdots], \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}\right\} \tag{4}$$

and therefore

$$x_k = x_k^{(1)} + (x_k^{(2)} - x_k^{(1)}) + (x_k^{(3)} - x_k^{(2)}) + \cdots \tag{5}$$

$$<< MC^k + PMC^{k+1} + P^2MC^{k+2} + \cdots = MC^k/(1 - PC). \tag{6}$$

The x_k as defined by (5) are a solution of (2), for if we add all the equations of (4) and sum by columns the resulting absolutely convergent double series, we have

$$x_k = c_k - [a_{kk+1}x_{k+1} + a_{kk+2}x_{k+2} + \cdots].$$

By (6) we see that for the x_k defined by (5) there exist μ and γ such that $|x_k| \leqq \mu\gamma^k$, $\gamma < (1/P)$, $\gamma \leqq 1$. Under this restriction the solution is unique, for if x_k' and x_k'' denote two solutions satisfying this restriction, their difference $\bar{x}_k = x_k' - x_k''$ is a solution of the homogeneous system corresponding to (2):

$$\left.\begin{aligned} \bar{x}_1 + a_{12}\bar{x}_2 + a_{13}\bar{x}_3 + \cdots &= 0, \\ \bar{x}_2 + a_{23}\bar{x}_3 + \cdots &= 0, \\ \bar{x}_3 + \cdots &= 0, \\ \cdots\cdots &, \end{aligned}\right\} \tag{7}$$

and we have

$$|\bar{x}_k| \leqq NX^k, \ X < (1/P), \ X \leqq 1.$$

Place

$$\begin{aligned} &x_k^{(1)} = \bar{x}_k \quad (k = 1, 2, \cdots), \\ &x_k^{(i+1)} = -[a_{kk+1}x_{k+1}^{(i)} + a_{kk+2}x_{k+2}^{(i)} + \cdots] \quad (i = 1, 2, \cdots). \end{aligned}$$

from which it follows that $|x_k^{(i+1)}| \leqq P^i N X^{k+i}$. Hence

$$\lim_{i=\infty} x_k^{(i)} = 0.$$

But we have $x_k^{(i)} = x_k^{(1)} = x_k$ by equations (7). Hence $x_k = 0$, which proves the uniqueness of the solution and completes the proof of Theorem I for the case $K = 0$. The reader will easily complete the proof of Theorem I in its generality $[K \neq 0]$ by the use of mathematical induction. In this proof, it will appear that equations (3) and (5) will give x_k for every value of k.

When the x_k defined by (3) and (5) are computed in terms of the a_{ij} and c_i, it is found that

$$x_k = c_k - \sum_{j=k+1}^{\infty} a_{kj} c_j + \sum_{j=k+1}^{\infty} \sum_{i=j+1}^{\infty} a_{kj} a_{ji} c_i - \cdots, \tag{8}$$

which is the so-called Neumann series.

The following special case of Theorem I will be used in the sequel:

Theorem II. *If for the system* (2) *we have* $\sum_{j=k+1}^{\infty} |a_{kj}|$ *convergent for every* k, $\sum_{j=k+1}^{\infty} |a_{kj}| \leqq P < 1$ *for every* $k > K$,

$$|c_k| \leqq C \textit{ for every } k,$$

then (2) *has one solution and only one solution for which the* x_k *are bounded. Moreover, this solution is given by the Neumann series* (8).*

We now return to the system of general type (1), and shall proceed to show that if (1) has a non-vanishing normal determinant and if the c_k are bounded, then (1) can be transformed into an equivalent system of type (2). This latter system will be shown to satisfy the hypotheses of Theorem II, and therefore the method of successive approximations can be used. To show the possibility of making this transformation we need the

Lemma. *In any non-vanishing determinant*

$$\Delta^{(k)} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix},$$

* Theorem II is similar to a theorem given by von Koch, *Jahresbericht*, 1913, p. 289.

the rows can be arranged so that no minor

$$\Delta^{(i)} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{vmatrix} \quad (i = 1, 2, \cdots k)$$

will vanish.

This lemma is evidently true (although trivial) for $k = 1$ and $k = 2$; the reader can readily complete the proof by induction.

We have supposed (1) to have a non-vanishing normal determinant; that is, we suppose $\sum_{i,j=1}^{\infty} | a_{ij} - d_{ij} |$ to be convergent [d_{ij} is the Kronecker symbol whose value is zero or unity according as $i \neq j$ or $i = j$] and

$$\lim_{n=\infty} \Delta^{(n)} = \Delta \neq 0.$$

Then there exists k such that $\Delta^{(n)} \neq 0$ for $n = k, k+1, \cdots$. Hence, by the Lemma, the order of the equations can be changed (if necessary) so that $\Delta^{(n)} \neq 0$ for $n = 1, 2, \cdots$. Such rearrangement will not affect the convergence of the double series $\sum_{i,j=1}^{\infty} | a_{ij} - d_{ij} |$ nor the value of Δ.

We suppose, now, that this arrangement has been made, and we proceed to transform (1) into an equivalent system of the type

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + \cdots &= \beta_1 , \\ b_{22}x_2 + b_{23}x_3 + \cdots &= \beta_2 , \\ b_{33}x_3 + \cdots &= \beta_3 , \\ \cdots\cdots & \end{aligned}\right\} \tag{9}$$

This transformation is made by placing* $b_{1k} = a_{1k}$, $\beta_1 = c_1$, and for $n > 1$,

$$b_{n,k} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{n,k} \end{vmatrix}, \quad (k = 1, 2, \cdots)$$

$$\beta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & c_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & c_n \end{vmatrix}.$$

The β_n as thus defined are bounded for if we set

* Riesz, *l. c.*, p. 11.

$$A_k = \sum_{i=1}^{\infty} | a_{ik} - d_{ik} |, \quad \Pi = \prod_{k=1}^{\infty} (1 + A_k),$$

we shall have

$$|\beta_n| = \text{absolute value of} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & c_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & c_n \end{vmatrix} \leqq \Pi \cdot C, \tag{10}$$

where $| c_k | \leqq C$ $(k = 1, 2, \cdots)$.

Similarly, we shall prove that $\sum_{j=i+1}^{\infty} | b_{ij} |$ converges for every i. For by using a method similar to that used in (10), it follows that for $n < k$,

$$| b_{nk} | \leqq \Pi \, [\, | a_{1k} | + | a_{2k} | + \cdots + | a_{nk} | \,].$$

Hence we have

$$\begin{aligned} &| b_{n,n+1} | + | b_{n,n+2} | + \cdots \\ &\leqq \Pi \, [\, | a_{1,n+1} | + | a_{1,n+2} | + \cdots \\ &\qquad + | a_{2,n+1} | + | a_{2,n+2} | + \cdots \\ &\qquad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\qquad + | a_{n,n+1} | + | a_{n,n+2} | + \cdots] . \end{aligned}$$

Then for every value of n, the series of absolute values of the coefficients b_{nk}, $k > n$, converges; and its sum approaches zero as n becomes infinite.

None of the b_{nn} is zero and they approach a limit different from zero, so when we write system (9) in the form (2) the hypothesis of Theorem II is satisfied. Therefore we have shown that if (1) has a non-vanishing normal determinant and the c_i are bounded, and if (1) is transformed into an equivalent system of type (2), then the method of successive approximations can be used. Incidentally we have proved the theorem of von Koch that if (1) has a non-vanishing normal determinant and if the c_i are bounded, then (1) has one solution and only one solution such that the x_i are bounded.

We shall now state two theorems, each of which is proved precisely as Theorem I was proved. The x_k of each theorem are given by the Neumann series (8).

Theorem III. *If system* (2) *is such that* $\sum_{j=k+1}^{\infty} | a_{kj} |^p$ $(p > 1)$ *converges for every* k, $\sum_{j=k+1}^{\infty} | a_{kj} |^p \leqq Q^p$ *for every* $k > K$,

$$| c_k | \leqq MC^k, \qquad C < \frac{1}{(1 + Q^{p/(p-1)})^{(p-1)/p}}$$

then (2) *has one solution and only one solution for which*

$$| x_k | \leqq \mu\gamma^k, \qquad \gamma < \frac{1}{(1 + Q^{p/(p-1)})^{(p-1)/p}}.\text{*}$$

Theorem IV. *Under the restrictions* $| a_{ik} | \leqq NT^{k-i}$ *for every* $k > i$,

$$| c_k | \leqq MC^k, \quad TC < 1/(1+N),$$

system (2) *has one solution and only one solution for which*

$$| x_k | \leqq \mu\gamma^k, \quad T\gamma < 1/(1+N).\dagger$$

MADISON, WIS.
June, 1917.

---

* A slightly less general theorem for the case $p = 2$ was proved by von Koch using infinite determinants. See *Proc. Camb. Cong. Math.* (1912) I, p. 354.

In proving Theorem III there will be found useful the following inequality due to Hölder:

$$|\sum_{k=1}^{\infty} a_k b_k |^p \leqq (\sum_{k=1}^{\infty} | a_k |^p) (\sum_{k=1}^{\infty} | b_k |^{\frac{p}{p-1}})^{p-1}.$$

See Riesz, *l. c.*, p. 45.

† Cf. von Koch, *l. c.*, p. 355.

Self-Dual Plane Curves of the Fourth Order.

By L. E. Wear.

§ 1. *Introduction.*

The reciprocal, r^m, of a plane rational curve, ρ^n, may be regarded as obtained by a polarity which sends any point of ρ^n into a line of r^m and conversely. The singularities of ρ^n will go over into their dual singularities on r^m. Now the reciprocal is, in general, distinct from the point-curve and $m \neq n$. The question naturally rises as to when will the two curves coincide. Curves having this property may be called self-dual.* It is evident that for curves of this kind there must be a one-to-one correspondence between the singularities of ρ^n and r^m. In other words the order and class must be the same and we have a necessary condition for self-duality expressed by the equation,

$$n = n(n-1) - 2d - 3c, \text{ i. e. } 2d + 3c = n(n-2).$$

In this paper the quartic curve is to be considered and the equation becomes, for $n = 4$,

$$2d + 3c = 8.$$

There are two solutions for this equation, viz.:

$$\text{I.} \quad d = 1, \ c = 2$$
$$\text{II.} \quad d = 4, \ c = 0$$

These are, respectively, the limaçon and the degenerate case of two conics. They will be considered in this order.

Part I. The Limaçon.

§ 2. *The Equation of the curve.*

The curve is symmetrical with respect to an axis which cuts it in the double point and in two other points which are the vertices of the curve. If we take, as triangle of reference, the axis of the curve and the tangents at the vertices, then the equation is

$$x_0 = at^4 - (a+2)t^2, \quad x_1 = (a-2)t^2 - a, \quad x_2 = (a-1)t^3 - (a+1)t. \quad (1)$$

* Appel, in *Nouvelles Annales de Math.*, XIII, p. 207, calls such curves "autopolaire." In that article he considered the problem of finding curves self-polar with regard to a given conic—the reverse of the present problem.

The Jacobians of these, two at a time, give the line equation of the curve, which is

$$
\begin{aligned}
\xi_0 &= (a-1)(a-2)\tau^2 - a(a+1), \\
\xi_1 &= a(a-1)\tau^4 - (a+1)(a+2)\tau^2, \\
\xi_2 &= 2a(2-a)\tau^3 + 2a(2+a)\tau.
\end{aligned} \qquad (2)
$$

That equations (1) are the equations of a limaçon may be seen as follows:

The curve is reflected into itself in the axis $x_2 = 0$, the reflexion being effected by the transformation,

$$t + t' = 0,$$

of which $t = 0$, $t = \infty$ are the double elements. These are the vertices of the curve. In addition to these two points, the line $x_2 = 0$ cuts out the parameters $t^2 = (a+1)/(a-1)$. If these are substituted in the equations of the curve they give only one point, which must, then, be a double point of the curve.

The fundamental involution,* i. e. a pencil of binary forms apolar to each of (1), is

$$
\begin{aligned}
&(a-2)t^4 + 4(a-1)t^3 + 6at^2 + 4(a+1)t + (a+2) \\
&+ \lambda\,[(a-2)t^4 - 4(a-1)t^3 \\
&+ 6at^2 - 4(a+1)t + (a+2)]
\end{aligned} \qquad (3)
$$

If we substitute the coefficients of this pencil in the condition for a cusp which is given by Professor Morley in his "Notes on Projective Geometry," p. 40, we find that the condition is satisfied, and, hence, that the curve has a cusp. Since the curve is reflected into itself in $x_2 = 0$, and since the cusp does not lie on the axis then there must be a second one, the reflexion of the first. The curve is, therefore, one having a double point and two cusps, i. e., is the limaçon.

The two cusps are given by $t^2 = 1$. The flexes are given by the Jacobian of the two members of the fundamental involution,† and are

$$t^2 = (a+1)(a+2)/(a-1)(a-2).$$

§ 3. *Polarities.*

Now any correlation which sends the curve into itself must interchange cusps and flexes. Hence there may be two such correlations corresponding to

* See a paper by Stahl, *Crelle*, Vol. 101, p. 300.

† Meyer: *Apolarität und Rationale Kurven*, p. 244.

the two ways in which the cusps and flexes may be paired. These two correlations are given by the transformations

$$t\tau = \sqrt{(a+1)(a+2)/(a-1)(a-2)} \tag{4}$$

and

$$t\tau = -\sqrt{(a+1)(a+2)/(a-1)(a-2)} \tag{5}$$

We require now that (4) and (5) shall send any point of the limaçon into a line of the curve and conversely. In order to do this operate with (4) or (5) on the equation of a point and identify the resulting expression in the parameter with the equation of a line. Thus will the parameter of a point of the curve be interchanged with that of a line of the curve and conversely, and hence there will be obtained a correlation which sends the curve into itself.

Now if we substitute in the incidence condition

$$(x\xi) = 0,$$

of point and line, the coördinates x_i from equations (1), we have the equation of any point of the limaçon; likewise, if we substitute the coördinates ξ_i from equations (2) we have the equation of any line tangent to the curve. Making these substitutions we find as the equations of point and line respectively,

$$\begin{aligned}&[at^4-(a+2)t^2]\,\xi_0+[(a-2)t^2-a]\,\xi_1\\&+[(a-1)t^3-(a+1)t]\,\xi_2=0\end{aligned} \tag{6}$$

and

$$\begin{aligned}&[(a-1)(a-2)\tau^2-a(a+1)]\,x_0+[a(a-1)\tau^4(a+1)(a+2)\tau^2]x_1\\&+[2a(2-a)\tau^3+2a(2+a)\tau]x_2=0\end{aligned} \tag{7}$$

If we now make the transformation (4) in equation (7), i. e. put $\tau = (1/t)\sqrt{(a+1)(a+2)/(a-1)(a-2)}$, clear the resulting equation of fractions and remove the factor $(a-1)$, the result is

$$\begin{aligned}&(a^2-1)(a-2)^2\sqrt{(a-1)(a-2)}\;[at^4-(a+2)t^2]x_0\\&+(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\;[(a-2)t^2-a]x_1\\&-2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\;[(a-1)t^3-(a+1)t]x_2=0.\end{aligned} \tag{8}$$

Since this is a line on the point t of the curve, it is identical with equation (6), regarding both equations as functions of the parameter. Making this identification, we have

$$\begin{aligned}-a\xi_0&=a(a^2-1)(a-2)^2\sqrt{(a-1)(a-2)}\;x_0,\\a\xi_1&=-a(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\;x_1,\\(a+1)\xi_2&=2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\;x_2,\end{aligned}$$

wherein we have equated the coefficients of t^4, t^0 and t respectively. Dividing the first equation by $-a$, the second by a and the third by $a+1$, the equations become finally,

$$\left.\begin{aligned}\xi_0 &= (1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\ x_0,\\ \xi_1 &= -(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\ x_1,\\ \xi_2 &= 2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\ x_2,\end{aligned}\right\}\tag{9}$$

If the transformation

$$\tau = -(1/t)\sqrt{(a+1)(a+2)/(a-1)(a-2)}$$

had been made in the equation of a line, then only ξ_2 will be changed in (9), since it comes from odd-powered terms, while ξ_0 and ξ_1 come from even powers. Furthermore the coördinate ξ_2 will be changed only in sign. Hence there arises from the second transformation the correlation

$$\left.\begin{aligned}\xi_0 &= (1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\ x_0,\\ \xi_1 &= -(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\ x_1,\\ \xi_2 &= -2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\ x_2,\end{aligned}\right\}\tag{10}$$

Equations (9) and (10) are two correlations which send any point of the curve into a line of the curve, i. e. send the curve into itself. In particular dual singularities are interchanged, as may be easily verified.

Furthermore if we examine the determinants of equations (9) and (10) we find them to be of the form

$$\begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c\end{vmatrix},$$

i. e. the two correlations are actually polarities and the conics giving them are

$$\begin{aligned}&(1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\ x_0^2\\ &-(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\ x_1^2\\ &\pm 2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\ x_2^2 = 0.\end{aligned}\tag{11}$$

These conics are conjugate, in the sense that they have double contact and are reflected, the one into the other, by a pair of reflexions in the reference triangle. It follows that each conic is its own polar reciprocal as to the other. This fact is seen from a geometrical viewpoint, since a polarity leaving the curve unaltered must also leave the conic of the other polarity unaltered.

The points of contact of the two conics lie on the axis $x_2 = 0$. This can be proved from general considerations as follows: Call the vertices of the

limaçon A and A' (See Fig. 1); the polar of A ($t = \infty$) is the tangent at $A'(t=0)$ and conversely. Hence A and A' are a pair harmonic to the meets of $x_2 = 0$ with both of the conics. Also the polar of the double point D is the double line which cuts the axis at D', say. Then D and D' are also a pair of the involution on the line and the Jacobian of the two pairs (A, A') and (D, D') will give the points (C, C') where the conics cut $x_2 = 0$ and where they have contact,—since the polars of those points are tangent to both conics at the poles themselves. Further, the line $x_2 = 0$ is an axis of each one of the conics, since the polars of points on it, (A, A', D, D'), are lines perpendicular to the axis of the limaçon, and the pole of the latter is a point at infinity where any two of these perpendiculars intersect.

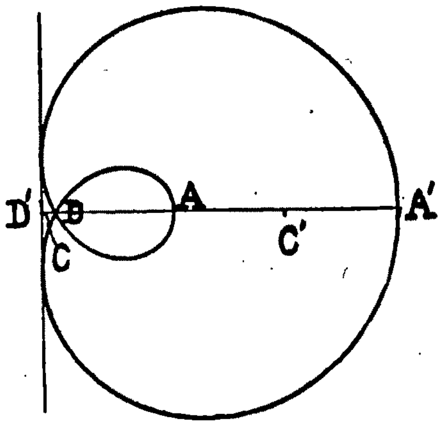


FIG. 1.

There will be certain points on the curve which will be fixed under the polarities, i. e. are transformed into tangents at the same points. These fixed points are four in number and are found by letting t and τ come together in equations (4) and (5). They are given by

$$t^4 = (a+1)(a+2)/(a-1)(a-2).$$

Since at these four points the polar of t is a tangent to the curve at t, therefore each of the conics (11) has double contact with the limaçon. One conic has real contacts with the curve and the other has imaginary contacts.

§ 4. *Special Cases.*

Conics (11) will degenerate when their discriminants vanish. The latter are $\pm 2a(a-1)(a+1)^4(a-2)^5(a+2)^5\sqrt{(a+1)(a+2)}$ which vanish for the values $a = \pm 1, \pm 2$. But equations (11) vanish identically

for $a = -1$ and $a = +2$. Since the flexes are given by

$$t^2 = (a+1)(a+2)/(a-1)(a-2),$$

the value $a = +1$ signifies that two flexes have united at $t = \infty$ and it may be easily verified that they unite to form a third cusp. The cusp-tangent is $x_2 = 0$, which, taken twice, is the equation of the conics for the value $a = -1$. For $a = -2$, two flexes of the curve unite but in this case they form an undulation point, i. e. a point where the tangent to the curve meets the curve

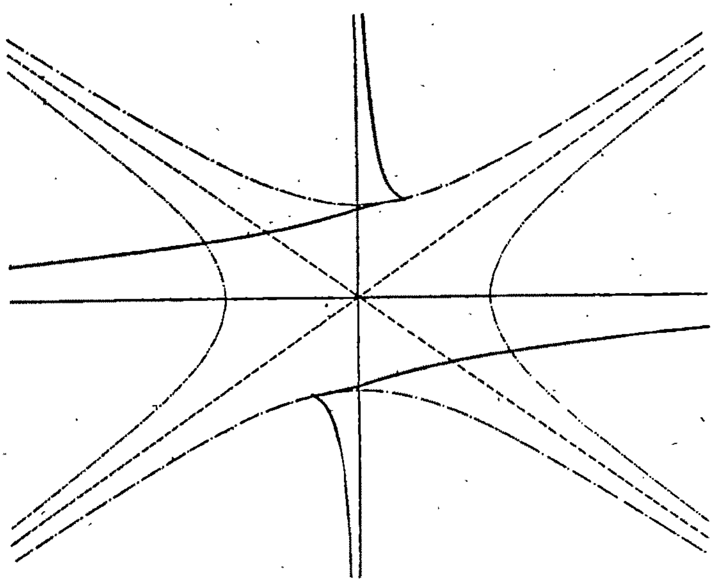


FIG. 2.

in four coincident points. The conics, for $a = -2$, degenerate into the line $x_0 = 0$, which is the equation of the undulation tangent.

§ 5. *Summary.*

We have seen that the limaçon admits of a reflexion given by the equation

$$t + t' = 0,$$

and is invariant under the two polarities π_0 and π_1. Call the reflexion R. Since $R^2 = 1$, the elements 1, R, form a group (G_2) of collineations under which the curve is invariant. It is evident also that

$$\pi_0 R = \pi_1 \text{ and } \pi_1 R = \pi_0.$$

Further

$$\pi_0\pi_1 = R \text{ and } \pi_0^2 = \pi_1^2 = 1.$$

Hence the

Theorem: *The limaçon is invariant under a G_4 consisting of two collineations and two polarities.*

Furthermore all possible polarities and correlations which leave the curve fixed are exhausted in π_0 and π_1. For, suppose another to exist—say π_m. Then either

$$\pi_0\pi_m = 1 \text{ or } \pi_0\pi_m = R$$

In the first case

$$\pi_m = \pi_0,$$

and in the second $\qquad \pi_m = \pi_1.$

In Figure 2 are shown the limaçon (in Cartesian coördinates), with the two conics, for the case $a = \frac{1}{2}$.

§ 6. *Satellite Conic.*

Of some interest in connecting with the plane quartic curve is the Satellite Conic of a line. In the case of the cubic curve the corresponding thing is the Satellite Line. Any line, ξ, will cut the cubic in three points. The tangents to the curve at these three points meet the curve again in three other points which lie on a line,* called the Satellite Line of ξ. In the case of the plane quartic a line, ξ, will cut the curve in four points T_i. The tangents to the curve at these four points meet the curve again in eight points which lie on a conic,† the Satellite Conic of the line ξ. The problem here is to find the actual equation of this conic for the limaçon.

The condition that three points of the plane rational quartic be on a line is as follows:

$$\begin{aligned} &p_{01}S_3^2 + p_{02}S_2S_3 + (p_{03} - p_{12})S_1S_3 + p_{12}S_2^2 + (p_{04} - p_{13})S_3 \\ &+ p_{13}S_1S_2 + (p_{14} - p_{23})S_2 + p_{23}S_1^2 + p_{24}S_1 + p_{34} = 0. \end{aligned}$$

The p_{ij} refer to the determinants $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ of the fundamental involution $(at)^4 + \lambda(bt)^4$ (See Art. 2), and the S_i are symmetric functions of the three parameters of the points. Substituting the appropriate values of the p_{ij} from

* Salmon, *Higher Plane Curves*, Art. 179.

† Salmon, *l. c.*, Art. 30.

the fundamental involution of Art. 2, we have, as the condition that three points of the limaçon be collinear,

$$-(a-1)(a-2)S_3^2-2(a^2-a-1)S_1S_3+a(a-1)S_2^2$$
$$+2(a^2+a-1)S_2-a(a+1)S_1^2+(a+1)(a+2)=0.$$

Herein set $t_1=t_2=T$, $t_3=t$ and we have

$$\begin{aligned}&-(a-1)(a-2)t^2T^4-2(a^2-a-1)(t+2T)(tT^2)\\&+a(a-1)(2tT+T^2)^2+2(a^2+a-1)(2tT+T^2)\\&-a(a+1)(t+2T)^2+(a+1)(a+2)=0.\end{aligned} \tag{13}$$

This is a relation, $f(T^4, t^2)=0$, connecting T, the point of tangency of a line, and the two remaining points of intersection, t. It says that, given the line T, the two remaining points of intersection with the curve are given by (13). Conversely, given any point t on the curve, there are four tangents, T, to the curve from this point, given by (13). Since *any* line T, on one of the cusps will be incident with a t, then (T^2-1) must be a factor of equation (13). Rearranging the latter in powers of T we have

$$\begin{aligned}&[(a-1)(a-2)t^2-a(a-1)]T^4-4tT^3-2[(a^2-a+1)t^2\\&-(a^2+a+1)]T^2+4tT+[a(a+1)t^2-(a+1)(a+2)]=0,\end{aligned} \tag{13'}$$

and dividing by (T^2-1) we have, finally,

$$\begin{aligned}&[(a-1)(a-2)t^2-a(a-1)]T^2-4tT\\&-[a(a+1)t^2-(a+1)(a+2)]=0.\end{aligned} \tag{13''}$$

We have here a relation, $f(T^2, t^2)=0$, connecting the point t and the line T of the limaçon. For the self-dual quartic, then, on a line T are two points t, and from a point t are two tangents T, i. e. the relation is a perfectly symmetric one. It is invariant under the reflexion, $t+t'=0$, and also under the transformations (4) and (5). Letting $t=T$, we have a quartic giving the four points of the curve where a point t is coincident with a point of tangency, T. This quartic is

$$(a-1)(a-2)T^4-2(a^2+2)T^2+(a+1)(a+2)=0,$$

which factors into,

$$(T^2-1)[(a-1)(a-2)T^2-(a+1)(a+2)]=0.$$

These two factors give the cusps and flexes, respectively. Evidently at these points the tangent lines have three coincident intersections with the curve. The incidence condition of point and line is $(x\xi)=0$. Put therein the values

of x_i from equations (1), obtaining

$$\begin{aligned}&[aT^4 - (a+2)T^2]\,\xi_0 + [(a-2)T^3 - a]\,\xi_1\\ &+ [(a-1)T^3 - (a+1)T]\,\xi_2 = 0.\end{aligned} \tag{14}$$

Arranging in powers of T we have

$$\begin{aligned}&a\xi_0 T^4 + (a-1)\xi_2 T^3 - [(a+2)\xi_0 - (a-2)\xi_1]T^2\\ &- (a+1)\xi_2 T - a\xi_1 = 0.\end{aligned} \tag{14'}$$

Given a line ξ, equation (14′) fixes the parameters of the four points of intersection with the curve.

Now let the T's of (14′), be the same as those of (13″); i. e. let the two equations have common roots. The condition that the two have common roots is the vanishing of their eliminant, which will be of the fourth degree in the coefficients of (13″), and of the second degree in those of (14′).* Hence a $f(\xi^2, t^8) = 0$, a relation connecting ξ and the eight points t when the points of tangency T_i lie on ξ. This eliminant, formed according to Sylvester's dialytic method,† is

$$\begin{aligned}
&a[a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2 + 2a(a+1)(a-1)(a-2)^2\\
&(a^2+2a-2)\xi_0\xi_1 + a(a-1)^2(a-2)^4\xi_1^2 - (a+1)^3(a-1)^3\\
&(a-2)\xi_2^2]t^8 - 8a(a+1)^2(a-1)(2a-1)\xi_0\xi_2 t^7 - 4[a(a+1)^2\\
&(a+2)(a^4+3a^3-4a+2)\xi_0^2 + 2a(a-2)\\
&(a^6+a^5-5a^4-3a^3+7a^2-2a-1)\xi_0\xi_1 + a^2(a-1)^3(a-2)^3\xi_1^2\\
&-(a+1)^2(a-1)^2(a^4-a^3-a^2-a+1)\xi_2^2]t^6 + 8[2(a+1)^2\\
&(2a-1)(a^2+a-1)\xi_0\xi_2 + (a-1)^3(a-2)(2a+1)\xi_1\xi_2]t^5\\
&+2[(a+1)^2(a+2)^2(3a^4+6a^3-4a+2)\xi_0^2 + 2(3a^8-21a^6\\
&+46a^4-28a^2+8)\xi_0\xi_1 + (a-1)^2(a-2)^2(3a^4-6a^3+4a+2)\xi_1^2\\
&-(a+1)^2(a-1)^2(3a^4-3a^2-4)\xi_2^2]t^4 - 8[(a+1)^3(a+2)\\
&(2a-1)\xi_0\xi_2 + 2(a-1)^2(2a+1)(a^2-a-1)\xi_1\xi_2]t^3\\
&-4[a^2(a+1)^3(a+2)^3\xi_0^2 + 2a(a+2)(a^6-a^5-5a^4+3a^3+7a^2+2a-1)\\
&\xi_0\xi_1 + a(a-1)^2(a-2)(a^4-3a^3+4a+2)\xi_1^2 - (a+1)^2(a-1)^2\\
&(a^4+a^3-a^2+a+1)\xi_2^2]t^2 + 8a(a+1)(a-1)^2(2a+1)\xi_1\xi_2 t\\
&+a[a(a+1)^2(a+2)^4\xi_0^2 + 2a(a+1)(a-1)(a+2)^2(a^2-2a-2)\\
&\xi_0\xi_1 + a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2 - (a+1)^3(a-1)^3\\
&(a+2)\xi_2^2] = 0.
\end{aligned} \tag{15}$$

Let a line ξ cut the curve in four points T. Draw the four tangents at these points. These tangents cut the curve in eight other points the para-

* See Salmon's Lessons on Higher Algebra, Art. 70.

† Salmon, l. c., Art. 83.

meters of which are given by the octavic (15). The next step is to find the equation of a conic passing through these eight points.

Any conic

$$\sum_{i,k=0}^{2} a_{ik}x_i x_k = 0, \qquad a_{ik} = a_{ki}$$

will cut the curve in eight points which are obtained by substituting in the equation of the conic the values of x_i from equations (1). We have then

$$\begin{aligned} &a_{00}[at^4-(a+2)t^2]^2+a_{11}[(a-2)t^2-a]^2+a_{22}[(a-1)t^3\\ &-(a+1)t]^2+2a_{01}[at^4-(a+2)t^2][(a-2)t_2-a]+2a_{02}[at^4 \qquad (16)\\ &-(a+2)t^2][(a-1)t^3-(a+1)t]+2a_{12}[(a-2)t^2-a]\\ &[(a-1)t^3-(a+1)t]=0.\end{aligned}$$

Simplifying and arranging in powers of t we have

$$\begin{aligned} &a^2a_{00}t^8+2a(a-1)a_{02}t^7+[-2a(a+2)a_{00}+(a-1)^2a_{22}\\ &\qquad\qquad\qquad\qquad +2a(a-2)a_{01}]t^6\\ &+[-2a(a+1)a_{02}-2(a-1)(a+2)a_{02}+2(a-1)(a-2)a_{12}]t^5\\ &+[(a+2)^2a_{00}+(a-2)^2a_{11}-2(a+1)(a-1)a_{22}-4(a^2-2)a_{01}]t^4\\ &+[-2(a+1)(a+2)a_{02}-4(a^2-a-1)a_{12}]t^3 \qquad (16')\\ &+[-2a(a-2)a_{11}+(a+1)^2a_{22}+2a(a+2)a_{01}]t^2\\ &+2a(a+1)a_{12}t+a^2a_{11}=0.\end{aligned}$$

This octavic in t gives the parameters of the eight points cut out by any conic. By identifying (16′) with the octavic of equation (15), we have more than enough conditions to determine the coefficients a_{ik} of the conic. On identifying the two we have:

Coefficients of t^8,

$$\begin{aligned} aa_{00}=&\,a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2+2a(a+1)(a-1)(a-2)^2\\ &(a^2+2a-2)\xi_0\xi_1+a(a-1)^2(a-2)^4\xi_1^2-(a+1)^3(a-1)^3\\ &(a-2)\xi_2^2;\end{aligned}$$

of t^7, $a_{02}=-4(a+1)^3(2a-1)\xi_0\xi_2$;

of t, $a_{12}=4(a-1)^2(2a+1)\xi_1\xi_2$;

of t^0, $$\begin{aligned} aa_{11}=&\,a(a+1)^2(a+2)^4\xi_0^2+2a(a+1)(a-1)(a+2)^2\\ &(a^2-2a-2)\xi_0\xi_1+a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2\\ &-(a+1)^3(a-1)^3(a+2)\xi_2^2;\end{aligned}$$

of t^4, $$\begin{aligned} -(a+1)(a-1)a_{22}-2(a^2-2)a_{01}=&\,2(a+1)^2(a+2)^2\\ &(a^4+2a^3+2a^2+4)\xi_0^2+4(a^8-5a^6+2a^4+18a^2-12)\xi_0\xi_1\\ &+2(a-1)^2(a-2)^2(a^4-2a^3+2a^2-4)\xi_1^2 \qquad (17)\\ &-2(a+1)^2(a-1)^2(a^4+a^2-4)\xi_2^2;\end{aligned}$$

of t^2, $(a+1)^2 a_{22} + 2a(a+2)a_{01} = -2a(a+1)^2(a+2)^3$

$$(a^2+2a+4)\xi_0^2 - 4a(a+2)(a^6-3a^4-4a^3+12a+6)\xi_0\xi_1 - 2a^4(a-1)^2(a-2)^2\xi_1^2 + 2(a+1)^2(a-1)^2 (a^4+2a^3+3a^2+2a-2)\xi_2^2 \quad (18)$$

By multiplying (17) by $(a+1)$ and (18) by $(a-1)$ and adding the resulting equations, we eliminate a_{22} and obtain

$$4a_{01} = 4(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 8(a^6-9a^4+18a^2-6)\xi_0\xi_1 + 4(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - 4(a+1)^2(a-1)^2(a^2-3)\xi_2^2,$$

or

$$a_{01} = (a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 2(a^6-9a^4+18a^2-6)\xi_0\xi_1 + (a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - (a+1)^2(a-1)^2(a^2-3)\xi_2^2.$$

Substituting this value of a_{01} in equation (18) and simplifying, we have

$$(a+1)^2 a_{22} = -4a(a+1)^4(a+2)^3\xi_0^2 - 8a^2(a+1)^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 - 4a(a+1)^2(a-1)^2(a-2)^3\xi_1^2 + 4(a+1)^2(a-1)^2(a^4+2a^3-2a-1)\xi_2^2$$

Hence,

$$a_{22} = -4a(a+1)^2(a+2)^3\xi_0^2 - 8a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 - 4a(a-1)^2(a-2)^3\xi_1^2 + 4(a-1)^2(a^4+2a^3-2a-1)\xi_2^2.$$

All the coefficients of the conics are determined. By substituting the proper values in the coefficients of t^6 and t^5 the conditions arising from those two terms are found to be satisfied by the above values. Hence the coefficients of the conic are the following:

$$aa_{00} = a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2 + 2a(a+1)(a-1)(a-2)^2 (a^2+2a-2)\xi_0\xi_1 + a(a-1)^2(a-2)^4\xi_1^2 - (a+1)^3(a-1)^3(a-2)\xi_2^2,$$

$$aa_{02} = -4a(a+1)^2(2a-1)\xi_0\xi_2,$$

$$aa_{12} = 4a(a-1)^2(2a+1)\xi_1\xi_2,$$

$$aa_{11} = a(a+1)^2(a+2)^4\xi_0^2 + 2a(a+1)(a-1)(a+2)^2 (a^2-2a-2)\xi_0\xi_1 + a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2 - (a+1)^3(a-1)^3(a+2)\xi_2^2,$$

$$aa_{01} = a(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 2a(a^6-9a^4+18a^2-6)\xi_0\xi_1 + a(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - a(a+1)^2 (a-1)^2(a^2-3)\xi_2^2,$$

$$aa_{22} = -4a^2(a+1)^2(a+2)^3\xi_0^2 - 8a^3(a+2)(a-2)(a^2-3)\xi_0\xi_1 - 4a^2(a-1)^2(a-2)^3\xi_1^2 + 4a(a-1)^2(a^4+2a^3-2a-1)\xi_2^2.$$

The satellite conic is then as follows:

$$
\begin{aligned}
&[a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2+2a(a+1)(a-1)(a-2)^2 \\
&(a^2+2a-2)\xi_0\xi_1+a(a-1)^2(a-2)^4\xi_1^2-(a+1)^3(a-1)^3 \\
&\qquad (a-2)\xi_2^2]x_0^2 \\
&+[a(a+1)^2(a+2)^4\xi_0^2+2a(a+1)(a-1)(a+2)^2 \\
&\qquad (a^2-2a-2)\xi_0\xi_1+a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2 \\
&\qquad -(a+1)^3(a-1)^3(a+2)\xi_2^2]x_1^2 \qquad\qquad (19) \\
&-4a[a(a+1)^2(a+2)^3\xi_0^2+2a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 \\
&\qquad +a(a-1)^2(a-2)^3\xi_1^2-(a+1)^3(a-1)^3\xi_2^2]x_2^2 \\
&+[8a(a-1)^2(2a+1)\xi_1\xi_2]x_1x_2-[8a(a+1)^2(2a-1)\xi_0\xi_2]x_0x_2 \\
&+2[a(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2+2a(a^6-9a^4+18a^2-6) \\
&\qquad \xi_0\xi_1+a(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2-a(a+1)^2(a-1)^2 \\
&\qquad (a^2-3)\xi_2^2]x_0x_1=0.
\end{aligned}
$$

Given a line ξ, cutting the limaçon in four points T: the tangents at the points T cut in eight other points which lie on the conic given in equation (19).

Among interesting special cases is that of the line $x_2=0$, the axis of the curve. Here $\xi_0=0$, $\xi_1=0$, $\xi_2=1$. Substituting these values in (19) we have, after simplifying,

$$
\begin{aligned}
&(a+1)^2[(a+1)(a-1)(a-2)x_0^2+2a(a^2-3)x_0x_1+(a+1) \\
&\qquad (a-1)(a+2)x_1^2]-4a(a+1)^3(a-1)x_2^2=0. \qquad (20)
\end{aligned}
$$

This is a conic symmetrical as to the line $x_2=0$ and passing through the double point. The latter fact can be seen from the following considerations: The line $x_2=0$ cuts the curve at the vertices and at the double point. The two tangents at the double point have there three points in common with the curve. Hence, two of the eight points, t_i, being at the double point, the Satellite must pass through the latter.

Again the line joining the two cusps is $x_0-x_1=0$. Putting $\xi_0=1$, $\xi_1=-1$, $\xi_2=0$ in (19) and simplifying, we have

$$
\begin{aligned}
&(2a-1)^4x_0^2+2(16a^4-8a^2-1)x_0x_1+(2a+1)^4x_1^2 \\
&\qquad\qquad -32a^2(2a+1)x_2^2=0. \qquad (21)
\end{aligned}
$$

This satellite of the line of cusps goes through the cusps themselves and through the two residual intersections of the cuspidal tangents.

Consider next the Satellite of a line ξ tangent to the curve. Such a line cuts in only three points T_i, the point of tangency and two others. The tangent lines at T_i will be the line ξ itself and the two tangents at other two points T_i. Call these tangents η and ζ. They cut the curve in t_1, t_2, and

t_1', t_2', respectively. Since ξ itself counts as a tangent at one of the points T_i, therefore the Satellite passes through the other points T_2, T_3. Since T_2, t_1, t_2 are on the conic, and likewise T_3, t_1', t_2', therefore the Satellite is composed of the two lines η and ζ. Hence, the Satellite conic of a line ξ tangent to the curve is composed of two lines, the tangents at the two points where ξ cuts the curve.

For example, the equation of the double line is

$$(a-2)^2 x_0 - (a+2)^2 x_1 = 0.$$

Substituting $\xi_0 = (a-2)^2$, $\xi_1 = -(a+2)^2$, $\xi_2 = 0$ in equation (19), we find the Satellite of the double line to be

$$(a-2)^4 x_0^2 - 2(a+2)^2(a-2)^2 x_0 x_1 + (a+2)^4 x_1^2 = 0, \tag{22}$$

or

$$[(a-2)^2 x_0 - (a+2)^2 x_1]^2 = 0. \tag{22'}$$

That is, the Satellite of the double line is the double line itself taken twice.

The equation of the flex tangent at

$$T_1 = \sqrt{(a+1)(a+2)/(a-1)(a-2)} \quad \text{is},$$

$$(a+1)(a-1)(a-2)^2 x_0 + (a+1)^2(a+2)^2 x_1 - 2a(a+2)(a-2)^2 \sqrt{(a+1)(a+2)/a-1)(a-2)}\ x_2 = 0. \tag{23}$$

Setting $\xi_0 = (a+1)(a-1)(a-2)^2$, $\xi_1 = (a+1)^2(a+2)^2$,
$\xi_2 = -2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)/(a-1)(a-2)}$

in (19), we have, as the Satellite,

$$\begin{aligned}(a+1)^2&(a-1)(a-2)^4(a^2+2a-1)x_0^2 + 2(a+1)(a+2)^2\\ &(a-2)^2(a^4+4a^2-1)x_0x_1 + (a+1)^2(a-1)(a+2)^4\\ &(a^2-2a-1)x_1^2 - 4a^2(a+1)(a+2)^3(a-2)^3 x_2^2\\ &-4a\sqrt{(a+1)(a+2)/(a-1)(a-2)}\ [(a-1)(a+2)^3\\ &(a-2)^2(2a+1)x_1x_2 - (a+1)(a+2)(a-2)^4\\ &(2a-1)x_0x_2] = 0.\end{aligned} \tag{24}$$

In this case there is only one intersection in addition to the flex point, namely

$$t = -\frac{(a-1)}{(a+1)}\sqrt{\frac{(a+1)(a+2)}{(a-1)(a-2)}}.$$

The tangent at the latter point is

$$\begin{aligned}(a+1)&(a-2)^2(a^2+2a-1)x_0 + (a-1)(a+2)^2(a^2-2a-1)x_1\\ &+ 2a(a-1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)/(a-1)(a-2)}\\ &x_2 = 0.\end{aligned} \tag{25}$$

The Satellite (24) is the product of the flex tangent itself, and the line (25).

Let us consider now the dual idea. From any point x in the plane four tangent lines, t_i, can be drawn to the curve, touching at four points. From each of these four points are two tangents, T_i, to the curve. By a process exactly analogous to the preceding it can be shown that the eight lines, T_i, lie on a conic, the Satellite of x. The relation $f(x_i, \xi_i) = 0$ so found is identical with equation (19). To prove this statement it is sufficient to say that, owing to the self-duality of the curve, the relation $f(\xi_i, x_i) = 0$, connecting a line ξ and its Satellite must be identical with the relation $f(x_i, \xi_i) = 0$ connecting a point x and its Satellite, since the two ideas are dual ones. Hence equation (19) has a dual interpretation: Given ξ, cutting the curve in four points, T_i, it is the equation of a conic on the eight points where tangents at T_i meet the curve again; given x, from which are four tangents, t_i, to the curve, it is the equation of a conic on the eight tangent lines of the curve drawn from the points of contact of the lines t_i.

The center of reflexion, admitted by the curve, furnishes a good illustration. The coördinates of the center (0, 0, 1) substituted in (19) give the Satelllite

$$\begin{aligned} a(a+1)^2(a+2)^3\xi_0^2 + 2a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 \\ + a(a-1)^2(a-2)^3\xi_1^2 - (a+1)^3(a-1)^3\xi_2^2 = 0. \end{aligned} \qquad (26)$$

This conic is on the double line and the four tangents drawn from the two vertices of the curve.

From a point x on the curve are only two tangents to the curve (t_1, t_2) in addition to the tangent at x. The lines t_1, t_2 will touch at y and z, say. From each of the points y and z are two tangents to the curve which lie on the Satellite of x. Furthermore, since the point x is the point of contact of one of the tangents *from* x (the tangent *at* x) therefore t_1 and t_2 lie on the Satellite. Hence, from each of the points y and z are three tangents to the conic, which must, therefore, break up into the two points y and z. Thus, the Satellite of a point x on the curve is composed of two points,—the points of contact of tangents from x.

For example the Satellite of the double point, $[(a+1), -(a-1), 0]$, is

$$[(a+1)\xi_0 - (a-1)\xi_1]^2 = 0, \qquad (27)$$

i. e., the square of the double point itself.

The coördinates of one of the cusps are 1, 1, 1. If substituted in (19), they give for the Satellite

$$(a+1)^2(a^2+2a-1)\xi_0^2+2(a^4+4a^2-1)\xi_0\xi_1+(a-1)^2 \\ (a^2-2a-1)\xi_1^2-(a+1)^2(a-1)^2\xi_2^2-2(a-1)^2 \\ (2a+1)\xi_1\xi_2+2(a+1)^2(2a-1)\xi_0\xi_2=0. \tag{28}$$

From the cusp there is only one tangent to the curve (in addition to the cuspidal tangent) and this meets the curve at $t=-(a+1)/(a-1)$. The equation of the latter point is

$$(a+1)^2(a^2+2a-1)\xi_0+(a-1)^2(a^2-2a-1)\xi_1 \\ -(a+1)^2(a-1)^2\xi_2=0, \tag{29}$$

and that of the cusp (1, 1, 1), is

$$\xi_0+\xi_1+\xi_2=0. \tag{30}$$

Equations (29) and (30) multiplied together give (28), i. e. the Satellite of the cusp consists of two lines one of which is the cusp tangent.

PART II. TWO CONICS.

§ 7. *The G_{24} of a Four-Point.*

It was pointed out in the introduction that only two cases of self-dual quartics are possible and we come now to the second case, that of two conics regarded as a degenerate ρ^4.

To study the properties of the curve it is necessary to consider the four-point common to the two conics and the group of collineations connected therewith; since, if the pair of conics is to be unaltered by correlations then their common four-point and four-line must be merely interchanged. The pair of conics intersect in the same four points, after being acted upon by the correlations, as they did before. In this sense the common four-point and therefore the common self-conjugate triangle are fixed. We assume then that the two conics have a proper, common, self-conjugate triangle, which is taken as the triangle of reference, and that the four points are in the canonical form $(1, \pm 1, \pm 1)$.

The four-point is invariant under a G_{24} of collineations, consisting of reflexions and collineations of periods three and four. Call the four points by the numerals 1, 2, 3, 4 and indicate by subscripts the interchanges made by the transformations. E. g., the notation $C_{(ij)(k)(l)}$ means a collineation interchanging i and j and leaving k and l fixed. Then, in the first place, there is a set of four reflexions (including identity) in the reference triangle, i. e. leaving the vertices of the reference triangle for centers and the sides for axes. In the notation just explained they are:

Firstly, $\qquad C_{(12)(34)},\ C_{(13)(24)},$ and $C_{(14)(23)}$.

These are of the type

$$x_0' = x_0, \quad x_1' = x_1, \quad x_2' = -x_2.$$

Secondly, there are reflexions

$$C_{(34)(1)(2)},\ C_{(14)(2)(3)},\ C_{(13)(2)(4)},\ C_{(24)(1)(3)},\ C_{(12)(3)(4)},\ C_{(13)(2)(4)},$$

which interchange two of the points and leave the other two fixed. E. g.

$$x_0' = x_1, \quad x_1' = x_0, \quad x_2' = x_2,$$

the center of which is $(1, -1, 0,)$ and the axis $x_0 - x_1 = 0$.

Thirdly, eight collineations of period three, leaving one point fixed and interchanging the other three cyclically

$$C_{(123)(4)},\ C_{(132)(4)},\ C_{(134)(2)},\ C_{(143)(2)},\ C_{(124)(3)},\ C_{(142)(3)},\ C_{(234)(1)},\ C_{(243)(1)}.$$

To illustrate, take the transformation

$$x_0' = x_2, \quad x_1' = x_0, \quad x_2' = x_1,$$

which is a member of the set.

Lastly, six collineations of period four, interchanging the four points cyclically

$$C_{(1234)},\ C_{(1324)},\ C_{(1342)},\ C_{(1243)},\ C_{(1423)},\ C_{(1432)}.$$

E. g. $x_0' = -x_2$, $x_1' = -x_1$, $x_2' = x_0$, is a member of this set.

The twenty-four collineations form a G_{24}, under which the four-point is invariant. Consider now the effects of these transformations on any conic passing through the four points $(1, \pm 1, \pm 1)$. Such a conic may be taken in the form

$$a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 = 0$$

provided that $$(a) = 0.$$

The reflexions 1, $C_{(ij)(kl)}$ involve only a change of signs and will, therefore, leave the conic unaltered. The other elements of the G_{24} in general send a conic on the four points into some other member of the pencil of conics determined by the four-point. Given, then, two conics considered as a quartic curve, the problem is to find correlations, and in particular, polarities, which will send any point of the curve into a line of the curve and conversely. This may be accomplished in either of two ways:

1°. The two conics may be interchanged under the correlations;

2°. Each conic may be unaltered.

§ 8. *General Case.*

Consider, first, the general case of two conics between which there exists no special relation. In general it is not possible to find a polarity that will leave each conic unaltered; but two conics related in a special manner do admit of such polarities and will be considered in the next article. It is necessary to look, then, for polarities that interchange the two conics.

Let

$$a_0x_0^2 + a_1x_1^2 + a_2x_2^2 = 0, \tag{31}$$

and

$$\beta_0x_0^2 + \beta_1x_1^2 + \beta_2x_2^2 = 0, \tag{32}$$

where $(a) = (\beta) = 0$, be two conics on the four points $(1, \pm 1, \pm 1)$. One of these is to be the polar reciprocal of the other as to some base conic. The latter may be taken in the form

$$ax_0^2 + bx_1^2 + cx_2^2 = 0; \tag{33}$$

since any polarity interchanging (31) and (32) will leave their common self-polar triangle unaltered and hence, either the latter is self-conjugate with respect to the base conic or else the base conic is tangent to two sides of the triangle, the third side being the chord joining the contacts. The latter possibility is considered in the next article. Now, if we require that the polarity as to (33) interchange (31) and (32) then the constants a, b, and c are determined to be

$$a : b : c : = \pm\sqrt{a_0\beta_0} \; : \pm\sqrt{a_1\beta_1} \; : \pm\sqrt{a_2\beta_2}$$

Take all possible combinations of signs and we obtain the following four polarities:

$$\begin{array}{llll}
\quad\pi_0 & \quad\pi_1 & \quad\pi_2 & \quad\pi_3 \\
\xi_0 = \sqrt{a_0\beta_0}x_0, & = -\sqrt{a_0\beta_0}x_0, & = \sqrt{a_0\beta_0}x_0, & = \sqrt{a_0\beta_0}x_0, \\
\xi_1 = \sqrt{a_1\beta_1}x_1, & = \sqrt{a_1\beta_1}x_1, & = -\sqrt{a_1\beta_1}x_1, & = \sqrt{a_1\beta_1}x_1, \\
\xi_2 = \sqrt{a_2\beta_2}x_2, & = \sqrt{a_2\beta_2}x_2, & = \sqrt{a_2\beta_2}x_2, & = -\sqrt{a_2\beta_2}x_2,
\end{array}$$

One of these, say π_0, having been obtained as above, then the others are given by the products $\pi_0 \cdot C_{(ij)(kl)}$. These are evidently correlations and they interchange the two conics, since π_0 interchanges them and $C_{(ij)(kl)}$ leave them fixed. That they are polarities appears from the fact that the products $\pi_0C_{(mn)(pq)} \cdot \pi_0C_{(rs)(tv)}$ which are collineations leaving each conic unaltered, must be contained in the set $C_{(ij)(kl)}$, and in particular $\pi_0C_{(ij)(kl)} \cdot \pi_0C_{(ij)(kl)}$ must be identity. Furthermore no other correlation could exist which would

transform the one conic into the other. For supposing such a one to exist, say π_m, then the product $\pi_0 \cdot \pi_m$ must be contained in the set $C_{(ij)(kl)}$, i. e.

$$\pi_0 \cdot \pi_m = C_{(mn)(pq)}$$

But

$$\pi_0 \cdot [\pi_0 \cdot C_{(mn)(pq)}] = C_{(mn)(pq)}$$

And hence

$$\pi_m = \pi_0 \cdot C_{(mn)(pq)}.^*$$

Hence the number of polarities is exhausted.

That there are just four polarities which interchange the cônics is evident from a geometrical point of view. For, the four points of intersection of the conics must go into their common lines. It would appear then that twenty-four polarities are possible; but any polarity sending one of the four common points into one of the four common tangents carries a unique transformation of the three other points into the three other tangents. Hence there are only four polarities possible.†

The elements 1, $C_{(ij)(kl)}$ form a G_4 of collineations under which the quartic is invariant. Furthermore it was shown above that all correlations leaving the curve unaltered are included in the set π_0, $[\pi_0 \cdot C_{(ij)(kl)}]$ and that the products of the latter, two at a time give $C_{(ij)(kl)}$.

Hence the Theorem: *A quartic curve, composed of two conics, is invariant under a G_8, consisting of four collineations and four correlations.*

§ 9. *Two Conics Subject to the Condition* $\Delta\theta_1^3 = \Delta_1\theta^3$.

We come next to the case of two conics admitting not only the polarities of the preceding article but also a second kind, viz. those which leave each conic separately unaltered.

Assume a pair of conics on the four points $(1, \pm 1, \pm 1)$, and such that either one of them is reflected into the other by one of the collineations $C_{(ij)(kl)}$,—say by the collineation

$$x_0' = x_0, \quad x_1' = x_2, \quad x_2' = x_1,$$

Such a pair are

$$a_0x_0^2 + a_1x_1^2 + a_2x_2^2 = 0, \tag{34}$$

$$a_0x_0^2 + a_2x_1^2 + a_1x_2^2 = 0. \tag{35}$$

---

* See Weber, Lehrbuch der Algebra, Vol. II, p. 4.

† On relations between two conics see Clebsch, Leçons sur La Géométrie, Vol. I, p. 150 *et seq.*

Furthermore, since

$$C_{(lm)(pq)} \cdot C_{(lm)(p)(q)} = C_{(pq)(l)(m)},$$

then the conics will be reflected into each other by a second member of the set $C_{(ij)(k)(l)}$.

Just as in the preceding case the curve is unaltered by the group 1, $C_{(ij)(kl)}$, π_0, $\pi_0 \cdot C_{(ij)(kl)}$, the polarities being as follows:

$$\begin{array}{llll}
\qquad\pi_0 & \qquad\pi_1 & \qquad\pi_2 & \qquad\pi_3 \\
\xi_0 = a_0x_0, & = -a_0x_0, & = a_0x_0, & = a_0x_0, \\
\xi_1 = \sqrt{a_1a_2}\, x_1, & = \sqrt{a_1a_2}\, x_1, & = -\sqrt{a_1a_2}\, x_1, & = \sqrt{a_1a_2}\, x_2, \\
\xi_2 = \sqrt{a_1a_2}\, x_2, & = \sqrt{a_1a_2}\, x_2, & = \sqrt{a_1a_2}\, x_2, & = -\sqrt{a_1a_2}\, x_2.
\end{array}$$

Now the reflexions $C_{(ij)(k)(l)}$ have the effect of interchanging the two conics, and $C_{(ij)(kl)}$ leave each unaltered. Hence the effect of the products $C_{(ij)(k)(l)} \cdot C_{(ij)(kl)}$ is to interchange the two. By these products we obtain four collineations which send each conic into the other. The four include the two reflexions $C_{(ij)(k)(l)}$ and two of the set $C_{(ijkl)}$, i. e. collineations of period four. The eight elements form a collineation G_8 under which the curve is invariant. Add now a polarity π_0 which interchanges the pair of conics. The products $\pi_0 \cdot C_{(ij)(kl)}$, as before, interchange the pair of conics; but the products $\pi_0 \cdot C_{(ij)(k)(l)}$ and $\pi_0 \cdot C_{(ijkl)}$, leave each conic fixed. Of these the first two are polarities and the other two correlations of period four. The four correlations are as follows:

$$\begin{array}{llll}
\multicolumn{2}{c}{\underline{\qquad\pi_0 \cdot C_{(ij)(k)(l)}\qquad}} & \multicolumn{2}{c}{\underline{\qquad\pi_0 \cdot C_{(ijkl)},\qquad}} \\
\xi_0 = a_0x_0, & = -a_0x_0, & = a_0x_0, & = a_0x_0, \\
\xi_1 = \sqrt{a_1a_2}\, x_1, & = \sqrt{a_1a_2}\, x_2, & = -\sqrt{a_1a_2}\, x_2, & = \sqrt{a_1a_2}\, x_2, \\
\xi_2 = \sqrt{a_1a_2}\, x_2, & = \sqrt{a_1a_2}\, x_1, & = \sqrt{a_1a_2}\, x_1, & = -\sqrt{a_1a_2}\, x_1.
\end{array}$$

It is readily seen that the determinants of the first two are symmetrical and that those of the other two are not. Hence the former are polarities while the latter are correlations. That the correlations are of period four may be easily verified and is evident from the fact that they are $\pi_0 \cdot C_{(ijkl)}$ which, raised to the fourth power, is $\pi_0^4 \cdot C^4_{(ijkl)}$, i. e. identity. Furthermore there can be no other correlations leaving the pair of conics unaltered, a fact easily proved just as in the preceding case. Hence the curve is invariant under a G_{16} of collineations and correlations. Of the latter six are polarities and two are of period four.

Consider now the base conics of the polarities $\pi_0 \cdot C_{(ij)(k)(l)}$. Their equations are

$$a_0x_0^2 \pm 2\sqrt{a_1a_2}\; x_1x_2 = 0. \tag{36}$$

These are of the nature of conjugate hyperbolas and may be called conjugate conics. Furthermore each of the original pair is conjugate to both of (36). What we have then is this: Two conics (31) and (32) which admit of a reflexion, the one into the other; two conics (36) each of which is conjugate to the other and is also conjugate to both (31) and (32).

We inquire now as to whether or not it was necessary to assume conics that admit of one of the reflexions $C_{(ij)(k)(l)}$ in order to obtain polarities whose base conics are conjugate to the original pair. Two conics admitting a reflexion may be taken in the forms

$$ax_0^2 + bx_1^2 + cx_2^2 \pm 2gx_0x_2 = 0.$$

The invariants, using Salmon's notation (Conic Sections, p. 334), are as follows:

$$\Delta = abc - bg^2, \quad \Delta_1 = abc - bg^2, \quad \theta = (3abc - bg^2), \quad \theta_1 = (3abc - bg^2).$$

Since an invariant relation must be homogeneous and of the same degree in the coefficients of both forms, the only relation subsisting between the invariants is

$$\Delta\theta_1^3 = \Delta_1\theta^3.$$

We seek next the condition that two conics have a common conjugate conic and may admit, therefore, of polarities such as $\pi_0 \cdot C_{(ij)(k)(l)}$. The pencil of conics having double contact with $(x^2) = 0$, and with $(x\xi) = 0$ as the common chord of contact is

$$(x\xi)^2 - \lambda[(x^2)(\xi^2) - (x\xi)^2] = 0.* \tag{37}$$

Values of λ that are equal but of opposite signs give conjugate pairs. For $\lambda = \pm 1$ we have

$$(x^2)(\xi^2) - 2(x\xi)^2 = 0 \tag{38}$$

and
$$(x^2) = 0 \tag{39}$$

Now a second line, $(x\eta) = 0$,

will determine another pencil

$$(x\eta)^2 - \mu[(x^2)(\eta^2) - (x\eta)^2] = 0 \tag{40}$$

and for $\mu = \pm 1$ the conjugate pair

$$(x^2)(\eta^2) - 2(x\eta)^2 = 0, \tag{41}$$

* See Salmon's Conic Sections, p. 340.

$$(x^2) = 0. \tag{42}$$

Hence (38) and (41) are both conjugate to $(x^2) = 0$. Writing $\sigma_1 = (\xi^2)$ and $\sigma_2 = (\eta^2)$, the invariants are

$$\Delta = -\sigma_1^3, \quad \Delta_1 = -\sigma_2^3, \quad \theta = \sigma_1\lambda, \quad \theta_1 = \sigma_2\lambda,$$

where

$$\lambda = [\sigma_1\sigma_2 - 4(\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2)^2].$$

The only invariantive relation subsisting between them is

$$\Delta\theta_1^3 = \Delta_1\theta^3.$$

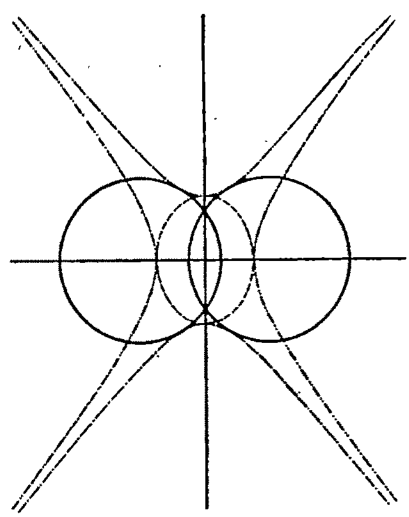


FIG. 3.

That is, the invariant relation that two conics admit of a reflexion the one into the other is identically the same as the condition that the two have a common conjugate conic. Hence the necessary and sufficient condition that polarities of the type $\pi_0 \cdot C_{(ij)(k)(l)}$ exist, is that the pair of conics admit of a reflexion and is expressed analytically by the above relation between the invariants.

We summarize this case by the Theorem: *Two conics characterized by the relation $\Delta\theta_1^3 = \Delta_1\theta^3$ are invariant under a G_{16} consisting of collineations and correlations. Of the latter six are polarities and two are of period four.*

In Fig. 3 the circles are the original pair of conics admitting a reflexion. Only three polarities are real, as shown in the figure.

§10. *The Clebschian Pair.*

We come finally to a special pair of conics which is invariant under the entire G_{24} of collineations and therefore, by adding a polarity and a G_{48} of collineations and correlations.

Suppose the conics of the preceding case are required to admit not only of two members of the set $C_{(ij)(k)(l)}$, as in that case, but also of a third member of the same set of collineations. Putting this further condition in the two conics we obtain the pair

$$x_0^2 + \omega x_1^2 + \omega^2 x_2^2 = 0, \quad x_0^2 + \omega^2 x_1^2 + \omega x_2^2 = 0. \tag{43}$$

These are either interchanged or left separately unaltered by the entire G_{24} of collineations. Hence, by adding a polarity, they are unaltered by a G_{48} of correlations and collineations. Of the former ten are polarities. Hence the Theorem: *The Clebschian pair of conics is invariant under a G_{48} of collineations and correlations. Of the latter ten are polarities and the others of periods three and four.*

On the Groups of Isomorphisms of a System of Abelian Groups of Order p^m and Type $(n, 1, 1, \cdots, 1)$.*

By Louis C. Mathewson.

Introduction.

Early in the study of groups of isomorphisms Moore showed that the group of isomorphisms of an abelian group of order p^m and type $(1, 1, \cdots, 1)$ is the linear homogenous group,† extensively discussed by Jordan in his *Traité des Substitutiones* (1870). Miller discussed the automorphisms of an abelian group of order p^m, type $(m-1, 1)$,‡ and later gave incidentally a formula for the order of the group of isomorphisms of any abelian group of order p^m.§ In 1907 Ranum through his study of the group of classes of congruent matrices showed that the group of isomorphisms of any given abelian group of order p^m was simply isomorphic with a certain chief n-ary linear congruence group.|| In the present paper the viewpoint is different and the groups are treated as abstract groups. The object is to study the groups of isomorphisms of the system of abelian groups of ordem p^m, type $(n, 1, \cdots, 1)$, $n > 1$, and to show that these groups of isomorphisms may be built upon the group of isomorphisms of an abelian group which contains no operators of order greater than p. To serve as a stepping stone to the general theory as well as to bring out the relations true for the first case, the case $n = 2$ will be considered immediately for $p = 2$ and for $p > 2$. In each development the group under consideration will be represented by G and its group of isomorphisms by I; p is used for an odd prime.

Theory.

Theorem 1. *The I of an abelian group of order 2^{m+1}, type $(2, 1, \cdots, 1)$ is of order $2^m(2^m-2)(2^m-2^2) \cdots (2^m-2^{m-1})$ and is simply isomorphic with a subgroup of index 2^m-1 in the holomorph of the abelian group of*

* Presented at the Dartmouth Meeting of the American Mathematical Society, Sept. 5, 1918.

† Cf. also Burnside, *Theory of Groups* (1897), §§ 171, 172 and Chap. XIV.

‡ Miller, *Transactions of the American Mathematical Society*, Vol. 2 (1901), pp. 259-264.

§ Miller, *Bulletin of the American Mathematical Society*, Vol. 20 (1913-14), p. 364.

|| Ranum, *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 71-91.

order 2^m, type $(1, 1, \cdots, 1)$. This I may be obtained by extending an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ by those operators from its own group of isomorphisms which leave one arbitrary operator in this abelian group fixed.

Suppose $m > 1$. The operators of order 2 in G evidently with the identity form a characteristic subgroup, H, of order 2^m. In H there is one (and only one) characteristic subgroup besides the identity. It is of order 2 and consists of the identity and the operator of order two which is the square of all the operators of order 4 in G. All the operators outside of H are of order 4. With the operators of H in identical correspondence any one of these operators may stand first, and an automorphism of G is then determined. Since these automorphisms are of order 2, commutative and number $2^m - 1$, the I of G contains an invariant abelian subgroup, H', of order 2^m, type $(1, 1, \cdots, 1)$. For, let $H \equiv 1, s_2, s_3, \cdots, s_h$, let $t^2 = s_2$, and let $G \equiv 1, s_2, s_3, \cdots, s_h, t, ts_2, \cdots, ts_h$ (all operators being commutative). Any operator ts_i, $i = 2, \cdots, h$, may correspond to t, so that the order of H' is h. Let $H' \equiv 1, v_2, v_3, \cdots, v_h$. Let the v that transforms t into ts_2 be v_2, into ts_3 be v_3, etc. Then, since each s is invariant under the v's

$$\begin{aligned} v_l^{-1}s_iv_l &= s_i, \\ v_l^{-1}tv_l &= ts_l. \end{aligned} \qquad l = 2, \cdots, h;\ i = 2, \cdots, h. \tag{1}$$

That the v's are of order 2 is evident from the fact that

$$v_l^{-1}(v_l^{-1}tv_l)v_l = v_l^{-1}ts_lv_l = v_l^{-1}tv_l \cdot v_l^{-1}s_lv_l = ts_l^2 = t;$$

and since all the operators of H' excepting the identity are of order 2, H' is abelian,* or,

$$\begin{aligned} v_a^{-1}v_b^{-1}tv_bv_a &= v_a^{-1}ts_bv_a = v_a^{-1}tv_a \cdot v_a^{-1}s_bv_a = ts_as_b, \\ v_b^{-1}v_a^{-1}tv_av_b &= v_b^{-1}ts_av_b = v_b^{-1}tv_b \cdot v_b^{-1}s\ v_b = ts_bs_a, \end{aligned}$$

and since $ts_as_b = ts_bs_a$, $v_a^{-1}v_b^{-1}tv_bv_a = v_b^{-1}v_a^{-1}tv_av_b$, or $v_av_b = v_bv_a$. That v_av_b transforms t into ts_as_b makes it possible to put H and H' into simple isomorphism in the following way: $s_i \sim v_i$, $i = 2, \cdots, h$.

From the nature of G, evidently H can be automorphic in all the ways an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ can, except that s_2 must always correspond to itself. This means that one subgroup of order 2 in H is always fixed, so that the order of the quotient group of the I of G with respect to the invariant H' as a head is equal to the order of the group of isomorphisms of H divided by $2^m - 1$.*

* Cf. Burnside, *loc. cit.*, p. 60.

* Burnside, *loc. cit.*, § 172.

It will now be shown that the I of G may be obtained by extending H' by operators which transform it in just the ways H may be transformed in G. This will be done by showing that an operator effecting any permissible automorphism of H, would produce a similar isomorphism among the operators of H'; i. e., if $u^{-1}s_k u = s_k$, then $u^{-1}v_k u = v_k$. It may be supposed that u is so chosen from the I of G that it transforms t into itself; for if not, u can be multiplied by such an operator from H' that the product will transform t into itself and at the same time effect exactly the same automorphism of H. Using (1),

$$(u^{-1}v_k u)^{-1}t(u^{-1}v_k u) = u^{-1}v_k^{-1}utu^{-1}v_k u = u^{-1}v_k^{-1}tv_k u = u^{-1}ts_k u =$$
$$u^{-1}tu \cdot u^{-1}s_k u = ts_k, \text{just as } v_k^{-1}tv_k = ts_k; \qquad (2)$$

and since the v's are commutative with the s's and so also is $u^{-1}v_l u$ (because $(u^{-1}v_l u)^{-1}s_i(u^{-1}v_l u) = u^{-1}v_l^{-1}us_i u^{-1}v_l u = s_i$, for $us_i u^{-1}$ is some s, and hence $v_l^{-1}us_i u^{-1}v_l = us_i u^{-1}$, and $u_l^{-1}us_i u^{-1}u = s_i$); therefore, $u^{-1}v_h u$ and v_k effect the same isomorphisms of G with itself. Thus, $u^{-1}v_k u = v_k$.

From the preceding it is obvious that the I of G is a subgroup of index $2^m - 1$ in the holomorph of the abelian group of order 2^m, type $(1, 1, \cdots, 1)$. This I should have exactly $\phi(4)$, or 2 invariant operators.* The operator besides the identity is easily shown to be v_2 according to the notation here used. (Note too that here v_2 corresponds to s_2 in an invariant subgroup of index 2^{m-1} in H). All the v's are commutative with v_2, and if in (2) $h = 2$ remembering that $u^{-1}s_2 u = s_2$), the result from the end of the preceding paragraph is $u^{-1}v_2 u = v_2$.

Next, the case in which p is an odd prime will be considered, and it will be shown that in this case the I of G is a direct product of two groups; and what these two groups are will be discussed.

The operators of order p in G evidently with the identity form a characteristic subgroup, J, of order p^m; also in J there is a characteristic subgroup, H, of order p whose $p - 1$ operators of order p are the pth powers of the operators of order p^2 in G. The operators of order p^2 correspond among themselves in every automorphism of G. With J in identical correspondence, any one of the p^m operators of order p^2 having the same pth power in H may stand first, and the automorphism of G is then fixed. Moreover, every such automorphism of G is of order p, and these $p^m - 1$ automorphisms of order p are commutative. These two facts may be proved just as similar facts were proved in the preceding case where the prime was 2. Let this invariant abelian subgroup of order p^m and type $(1, 1, \cdots, 1)$ in the I of G be E, and let its oper-

* Miller, Blichfeldt, and Dickson, *Finite Groups* (1916), p. 162.

ators be v's. If the generators of J are $s_1, s_2, \cdots, s_m$ where s_1 generates H (and $t^p = s_1$), then by a method analogous to that used for the even prime 2, it can be shown that the correspondence between J and E can be taken as $s_i \sim v_i$ $(i = 1, \cdots, m)$, where $v_i^{-1}tv_i = ts_i$ and where $v_i^{-1}s_lv_i = s_l$ $(i = 1, \cdots, m; l = 1, \cdots, m)$.

From the nature of G, evidently J can be automorphic in all the ways an abelian group of order p^m, type $(1, 1, \cdots, 1)$ can, excepting that H must always correspond to itself. Since J has $(p^m - 1)/(p - 1)$ subgroups of order p,* this means that the order of the quotient group of the I of G with respect to the invariant E is equal to the order of the group of isomorphisms of J divided by $(p^m - 1)/(p - 1)$, which gives the order of the I of G as $p^m(p-1)(p^m - p)(p^m - p^2) \cdots\cdots (p^m - p^{m-1})$.

Consider those automorphisms of G in which the operators of H are in identical correspondence. Suppose that the operator, u, effecting the isomorphism under consideration transforms t into itself; for if not, u can be multiplied by such an operator from E that the product transforms t into itself and at the same time effects exactly the same automorphism of J. As in the preceding theorem, it can be shown easily that u transforms the operators of E among themselves in the same way it transforms the corresponding operators of J; that is, if

$$u^{-1}s_1u = s_1, \quad u^{-1}s_ju = s_{j'}, \quad (j = 2, \cdots, m; \; j' = 2, \cdots, m);$$

then $u^{-1}v_1u = v_1$, $u^{-1}v_ju = v_{j'}$, (j and j' have the same values respectively before).

Since all the automorphisms of G with the operators of the characteristic subgroup H in identical correspondence have been considered, the I of G evidently contains an invariant subgroup I' which is simply isomorphic with an abelian group of order p^m, type $(1, 1, \cdots, 1)$ extended by those operators from its own group of isomorphisms that leave the operators of one and only one of its subgroups of order p fixed. Since all other automorphisms of G arise from the automorphisms of H, and H is cyclic and of order p, obviously the quotient group of I with respect to I' is a cyclic group of order $p - 1$. It will now be shown that the I of G is simply isomorphic with the direct product of I' and the cyclic group of order $p - 1$.

The central of the I of G is the group of totitives (mod p^2) of order $\phi(p^2) = p(p-1)$; that is, a cyclic group, the product of a cyclic group of order p by a cyclic group of order $p - 1$. Each of these operators in the central of the I of G transforms every operator of G into the same power of

* Burnside, *loc. cit.*, p. 59.

itself.* The cyclic group of order p in this central has been already obtained. In the notation here used, it is generated by v_1, since $v_1^{-1}tv_1 = ts_1 = t^{1+p}$, $v_1^{-2}tv_1^2 = ts_1^2 = t^{1+2p}$, etc.; and since, for any s, $v_1^{-1}s_iv_1 = s_i$ $(i = 1, \cdots, m)$, and the $(1+kp)$th power of s_i is s_i; and since $u^{-1}v_1u = v_1$. If now $s_1 \sim s_1^n$, then since $t^p = s_1$, t can $\sim t^n$, and the remainder of the automorphism of G may be set up by having the other generators (besides s_1) of J correspond to their own nth powers, $s_i \sim s_i^n$ $(i = 2, \cdots, m)$. If w effects this automorphism of G, $w^{-1}s_iw = s_i^n$ $(i = 1, \cdots, m)$, $w^{-1}tw = t^n$ (also $w^{-1}s_i^aw = s_i$, $w^{-1}t^aw = t$), it is necessary and sufficient to show that w is commutative with all the v's and with u. First, it will be shown that $w^{-1}v_iw = v_i$. Since $v_i^{-1}tv_i = ts_i$ $(i = 1, \cdots, m)$, $(w^{-1}v_iw)^{-1}t(w^{-1}v_iw) = w^{-1}v_i^{-1} \cdot wtw^{-1} \cdot v_iw = w^{-1} \cdot v_i^{-1}t^av_i \cdot w = w^{-1}(v_i^{-1}tv_i)^aw = w^{-1}(ts_i)^aw = w^{-1}t^aw \cdot w^{-1}s_i^aw = ts_i$, just as $v_i^{-1}tv_i = ts_i$, and since the v's are commutative with s_j $(j = 1, \cdots, m)$ and so also is $w^{-1}v_iw$ (because $(w^{-1}v_iw)^{-1}s_j(w^{-1}v_iw) = w^{-1}v_i^{-1} \cdot ws_jw^{-1} \cdot v_iw = w^{-1}v_i^{-1}s_j^av_iw = w^{-1}s_j^aw = s_j$); therefore, these two operators from I are identical or $w^{-1}v_iw = v_i$. Second, to show that w is commutative with u, use will be made of

$$\begin{cases} u^{-1}s_1u = s_1, \\ u^{-1}s_ju = s_{j'} \end{cases} \quad (j = 2, \cdots, m;\ j' = 2, \cdots, m) \text{ and } u^{-1}tu = t. \text{ Here}$$

$$(w^{-1}uw)^{-1}s_j(w^{-1}uw) = w^{-1}u^{-1} \cdot ws_jw^{-1} \cdot uw = w^{-1}u^{-1}s_j^auw \quad (j = 1, \cdots, m)$$

$$= \begin{cases} \text{if } j = 1,\ w^{-1}s_1^aw = s_1, \text{ just as } u^{-1}s_1u = s_1. \\ \text{if } j = 2, \cdots, m,\ w^{-1}s_{j'}^aw = s_{j'}, \text{ just as } u^{-1}s_ju = s_{j'}; \end{cases}$$

also $(w^{-1}uw)^{-1}t(w^{-1}uw) = w^{-1}u^{-1} \cdot wtw^{-1} \cdot uw = w^{-1}u^{-1}t^auw = w^{-1}t^aw = t$.

Hence, not only is the quotient group of I with respect to I' the cyclic group of order $p-1$, but I contains such a cyclic group whose operators (excepting the identity) lie outside of I' and are commutative with each of the operators of I'. Therefore, I is simply isomorphic with the direct product of I' and the cyclic group of order $p-1$, and for $m > 1$ there results the

Theorem 2. *The I of an abelian group of order p^{m+1}, type $(2, 1, \cdots, 1)$ is of order $p^m(p-1)(p^m-p)\cdots(p^m-p^{m-1})$ and is simply isomorphic with the direct product of a cyclic group of order $p-1$ and the group formed by extending an abelian group of order p^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave the operators of some one of its subgroups of order p in identical correspondence.*

As a side step from the main problem of this paper the following general

* Miller, *Transactions of the American Mathematical Society*, Vol. 2 (1901), p. 260.

proposition concerning a property obtaining in the abelian group just considered, will now be discussed.

Subsidiary Theorem. *If an abelian group G contains a characteristic subgroup H of prime order p, the I of G is simply isomorphic with a direct product, one factor of which is the cyclic group of order $p-1$. This I is then divisible if $p>2$.*

Since "every abelian group is the direct product of its Sylow subgroups whenever its order is the product of more than one different prime," * H is in the Sylow subgroup, S, of order p^m and type $(m_1, m_2, \cdots, m_l)$, $m_1 \geqq m_2 \geqq \cdots \geqq m_l$, where $m = m_1 + m_2 + \cdots + m_l$ and $m_1 > 0$. If m_1 is not greater than m_2, there is no characteristic subgroup of order p. If $m_1 > m_2$, the group of order p in the cyclic group of order p^{m_1} is a characteristic subgroup of G (and the only characteristic subgroup of order p), since its operators are the p^{m_1-1}th powers of all the operators of S, its operators of order p being the p^{m_1-1}th powers of the operators of order p^{m_1} in S. Incidentally then, it has been shown that *a necessary and sufficient condition that an abelian group of order p^m contain a characteristic subgroup of order p is that there be one and only one largest invariant.* This group, H, of order p is a fundamental characteristic subgroup.† The remainder of the proof will now be worked out with respect to S, since the I of G is the direct product of the groups of isomorphisms of its Sylow subgroups.

The operators effecting the automorphisms of S in which the operators of H remain in identical correspondence form an invariant subgroup, I', of the group of isomorphisms of S (I_S). The quotient group of I_S with respect to I' is the group of isomorphisms of H; i. e., the cyclic group of order $p-1$. In the notation here used and with $p>2$, the central of I_S is a cyclic group of order $\phi(p^{m_1}) = p^{m_1-1}(p-1)$, the product of a cyclic group of order p^{m_1-1} and another cyclic group of order $p-1$, and each of these operators in the central of I_S transforms every operator of S into the same power of itself.‡ The cyclic group of order p^{m_1-1} is in I', its operators being those which transform operators of order p^{m_1} in S into their $(1+kp)$th powers, $k=1, 2, \cdots, p^{m_1-1}$. The operators of order p in H are invariant individually under such transformations, since these powers of each opeartor of order p in H are that operator itself. The operator of I_S which transform every operator of S into the 2nd, 3d, $\cdots$, $(p-1)$th powers of itself, evidently transform

* Miller, Blichfeldt, and Dickson, *loc. cit.*, p. 87.

† Cf. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVII (1905), p. 15; also Miller, Blichfeldt, and Dickson, *loc. cit.*, p. 110.

‡ Miller, *Transactions of the American Mathematical Society*, Vol. 1 (1900), p. 397; Vol. 2 (1901), p. 260.

the operators of H among themselves, so are not in I'. They with the identity, constitute a cyclic group of order $p-1$, and since they are in the central of I_g; they are individually commutative with the operators of I'. Hence, since the quotient group of I_g with respect to I' is a cyclic group of order $p-1$, and I_g contains a cyclic group of order $p-1$ having only the identity in common with I' and each of its operators is commutative with each of the operators of I', therefore I_g is simply isomorphic with the direct product of I' and a cyclic group of order $p-1$.

The theory of the second theorem will now be extended to any abelian group, G, of order p^{m+n-1}, type $(n, 1, 1, \cdots (m-1)$ units), $n > 1$, p a prime > 2. Let t be the operator of order p^n, and $s_2, \cdots, s_m$ independent generators of order p, and for convenience let $t^{p^{n-1}} = s_1$. From the observations under the Subsidiary Theorem, s_1 generates a characteristic cyclic subgroup, $H^{(n)}$, of order p. All the operators of order p in G form a characteristic abelian subgroup $H^{(n-1)}$ of order p^m, type $(1, \cdots, 1)$. $H^{(n)}$ is in $H^{(n-1)}$. Then there is a characteristic abelian subgroup, $H^{(n-2)}$, of order p^{m+1}, type $(2, 1, \cdots, 1)$ generated by the $p^{(n-2)}$th powers of operators of order p^n and $H^{(n-1)}$. So likewise, $H^{(n-3)}$ is generated by the p^{n-3}th powers of operators of order p^n and $H^{(n-1)}$; and in general, $H^{(n-r)}$, of order p^{m+r-1} and type $(r, 1, \cdots, 1)$, is generated by the p^{n-r}th powers of the operators of order p^n and $H^{(n-1)}$, $r = 2, \cdots, n-1$. Each of these characteristic subgroups contains the preceding, and the largest $(H^{(1)})$ is of index p in G itself. This series of subgroups forms a characteristic series of G.*

Now with the operators of $H^{(1)}$ in identical correspondence, evidently t can correspond to any one of $p^m - 1$ operators besides itself; it can correspond to itself multiplied by any one of the operators of $H^{(n-1)}$, since these products (and t) alone are of order p^n and have the same pth power that t has in $H^{(1)}$. As was shown in connection with Theorem 2, these isomorphisms (excepting the identity) are of order p and commutative. Hence, the I of G contains an invariant abelian subgroup, E, of order p^m, type $(1, \cdots, 1)$, which is simply isomorphic with $H^{(n-1)}$; and, moreover, if the same convenient notation be employed here as in the theorem to which reference has just been made, the correspondence, $v_i \sim s_i$, $i = 1, \cdots, m$, can be set up, where the v's are the independent generators of the subgroup E. The transformations, accordingly, are $v_i^{-1}tv_i = ts_i$, $(i = 1, \cdots, m)$, $v_i^{-1}s_jv_i = s_j$, $(i = 1, \cdots, m; j = 1, \cdots, m)$.

Now, let the operators of $H^{(2)}$ be in identical correspondence. t^p can

* Frobenius, *Berliner Sitzungsberichte* (1895), p. 1027; cf. Burnside, *loc. cit.*, §§ 163, 164.

correspond to $t^{p^{n-1}+p}$, $t^{2p^{n-1}+p}, \cdots, t^{(p-1)p^{n-1}+p}$ (or t^ps_1, $t^ps_1^2, \cdots, t^ps_1^{p-1}$, respectively, since $t^{p^{n-1}}=s_1$) besides itself, because these and these alone have the same pth power in $H^{(2)}$ and at the same time are themselves the pth powers of operators outside $H^{(1)}$. Simultaneously with $t^p \sim t^{p^{n-1}+p}$, t must correspond to some operator of order p^n whose pth power is $t^{p^{n-1}+p}$; such operators are $t^{p^{n-2}+1}$ times operators from $H^{(n-1)}$. The operators from $H^{(n-1)}$ may be supposed to be in $H^{(n)}$ also, for if not, the operator effecting the automorphism of G under consideration can be multiplied by such a v that $H^{(1)}$ is transformed as stated in the preceding and t corresponds to $t^{p^{n-2}+1}$ times some operator from $H^{(n)}$. Accordingly, this isomorphism of G may be said to be effected by an operator which transforms every operator into its $(kp^{n-2}+1)$th powers, $k=1, \cdots, p-1$.

Similarly, if the operators of $H^{(3)}$ are in identical correspondence, the additional isomorphisms of G spring from those in which the operators correspond to their $(kp^{n-3}+1)$th powers, $k=1, \cdots, p-1$.

More generally, if the operators of $H^{(r)}$, $r=2, \cdots, n-1$, are in identical isomorphism, $t^{p^{r-1}}$ can correspond to $t^{p^{n-1}+p^{r-1}}$, $t^{2p^{n-1}+p^{r-1}}$, $\ldots, t^{(p-1)p^{n-1}+p^{r-1}}$ (or $t^{p^{r-1}}s_1, \cdots, t^{p^{r-1}}s_1^{p-1}$, respectively) besides itself, because these operators and these only have the same pth power in $H^{(r)}$ and are themselves the p^{r-1}th powers of operators outside $H^{(1)}$. These isomorphisms are those effected by an operator which transforms the operators of G into their $(kp^{n-r}+1)$th powers, $k=1, \cdots, p-1$. These automorphisms are p^{n-2} in number (because $r=2, \cdots, n-1$ and $k=1, \cdots, p-1$). If when $r=1$, v_1 is included, these isomorphisms number p^{n-1}, and they are, moreover, those in which the operators of G go over into their $(1+kp)$th powers, $k=1, 2, \cdots, p^{n-1}$. The only other powers are the 1st, 2nd, $\cdots$, $(p-1)$th, and when these are effected the characteristic subgroup $H^{(n)}$ takes all its automorphisms. But the operators effecting the possible transformations of all the operators into their same powers constitute the central of the I of G, a cyclic group of order $\phi(p^n)=p^{n-1}(p-1)$, (because the highest order of operators in G is p^n), the product of two cyclic groups, one of order $p-1$ and one of order p^{n-1}. From the Subsidiary Theorem the I of G is the direct product of this cyclic group of order $p-1$ and another subgroup, I'. The cyclic group of order p^{n-1} must be in I'. Suppose u to be employed to represent a generator of this cyclic group, F, of order p^{n-1}, so that $u^{-1}tu=t^{1+kp}$, $k=1, \cdots, p^{n-1}$. With the operators of $H^{(n-1)}$ in identical correspondence, all the possible isomorphisms of G are effected by u and the v's. They all are commutative (since u is in the central of I) and the cross-cut of E and F is the cyclic group of order p generated by $v_1\,(=u^{p-2})$.

Hence, I' contains an invariant abelian subgroup of order p^{m+n-2}, type $(n-1, 1, 1, \cdots, 1)$.

Finally, if the operators of $H^{(n)}$ alone are in identical correspondence, the remaining operators of $H^{(n-1)}$ have exactly the automorphisms of an abelian group of order p^{m-1}, type $(1, 1, \cdots, 1)$ when the operators of some one subgroup of order p remain fixed. If w effects such an automorphism (and leaves t invariant), it can be shown just as in Theorem 2 that w is commutative with v_1 but transforms the other operators of E just as the operators of $H^{(n-1)}$ outside of $H^{(n)}$ are transformed, and w, furthermore, would be found to be commutative with u. The number of the isomorphisms effected by w's would be $(p^m - p)(p^m - p^2) \cdots\cdots (p^m - p^{m-1})$.*

The following may be stated as a summary of these results:

Theorem 3. *The I of an abelian group G of order p^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, p a prime > 2, $n > 2$, is of order $(p-1)p^{m+n-2}(p^m - p)(p^m - p^2) \cdots\cdots (p^m - p^{m-1})$, and is simply isomorphic with the direct product of a cyclic group of order $p-1$ and a group formed by extending an abelian group of order p^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave invariant the operators of one its own group of isomorphisms which leave invariant the operators of one cyclic group of order p, and then multiplying this extended group by an operator of order p^{n-1} which is commutative with each operator of the extended group and which has one of the invariant operators of order p for its p^{n-2}th power.*

This shows that *for a given odd prime p and a fixed value of $m > 0$, the group of isomorphisms of each abelian group of the system $(n = (2), 3, 4, \cdots,)$ contains the group of isomorphisms of the preceding as an invariant subgroup of index p,* since they differ only in the order of the operator by which the extended group is multiplied.

Again, since multiplying the extended group by the designated operator of order p^{n-1} is equivalent to taking an abelian group of order p^{m+n-2}, type $(n-1, 1, \cdots, 1)$ and extending it by all those operators from its own group of isomorphisms that leave invariant the operators of exactly one of its subgroup of order p^{n-1}, the preceding theorem can be stated as follows, and from it can be seen that *the group of isomorphisms of each abelian group of the system under study is an extension of the one of the system just before it and of index p in it.*

Theorem 3'. *The I of an abelian group G of order p^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, p an odd prime and $n > 1$, is of order $(p-1)p^{m+n-2}(p^m - p)$*

* Cf. Burnside, *loc. cit.*, § 48.

$(p^m - p^2) \cdots (p^m - p^{m-1})$ and is simply isomorphic with the direct product of a cyclic group of order $p - 1$ and the group formed by extending an abelian group of order p^{m+n-2}, type $(n-1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms that leave invariant the operators of exactly one of its cyclic subgroups of order p^{n-1}.

If p is the even prime and $n > 2$, F is not cyclic but is an abelian group of order 2^{n-1}, type $(n-2, 1)$,* and the 2^{n-3}th power of the operators of order 2^{n-2} generates the cross-cut of F and E, a group of order two. Accordingly, the counterparts of the preceding two theorems are:

Theorem 4. *The I of an abelian group G of order 2^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, $n > 2$, is of order $2^{m+n-2}(2^m - 2)(2^m - 2^2) \cdots (2^m - 2^{m-1})$ and is simply isomorphich with the direct product of a group of order 2 and a group formed by extending an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave invariant one operator of order two, and then multiplying this extended group by an operator of order 2^{n-2} which is commutative with each operator of the extended group and which has the invariant operator of order two for its 2^{n-3}th power.*

This and the following equivalent statement of the proposition show the inclusive relation between the groups of isomorphisms of two consecutive groups of the system:

Theorem 4. *The I of an abelian group of order 2^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, $n > 2$, is of order $2^{m+n-2}(2^m - 2)(2^m - 2^2) \cdots (2^m - 2^{m-1})$ and is simply isomorphic with the direct product of a group of order 2 and the group formed by extending an abelian group of order 2^{m+n-3}, type $(n-2, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms that leave invariant individually the operators of exactly one of its cyclic subgroups of order 2^{n-2}.*

DARTMOUTH COLLEGE,
Hanover, New Hampshire.

---

* Burnside, *loc. cit.*, § 169.

On the Satellite Line of the Cubic.

By R. M. Winger.

The Rational Cubic.

1. *Introduction.* While Salmon * discusses the satellite line of the cubic at some length he fails to give its equation. The equation was supplied by Cayley † who calculated it laboriously by a direct method. Several covariant expressions for this remarkable line were given by Walker.‡ And another has been furnished by Morley.§ In the present paper the explicit equation of the satellite is exhibited both for the rational and the general cubic, in canonical form, and some associated loci are considered. Several interesting chain theorems are obtained and a generalization is made for the plane curve of order n.

We consider first the rational cubic R_4^3, taking as its equations in points and lines respectively

$$x_0 = 3t^2, \quad x_1 = 3t, \quad x_2 = t^3 + 1, \tag{1}$$

or

$$x_0^3 + x_1^3 - 3x_0x_1x_2 = 0,$$

$$u_0 = 1 - 2t^3, \qquad u_1 = t^4 - 2t, \qquad u_2 = 3t^2, \tag{2}$$

or

$$u_2^4 - 6u_0u_1u_2^2 - 3u_0^2u_1^2 + 4u_0^3u_2 + 4u_1^3u_2 = 0,$$

when the triangle of reference is the invariant triangle of the G_6 which leaves the curve unaltered.

An important curve for the satellite theory is the conic N, || the locus of lines joining pairs of contacts of tangents from points of the R_4^3. The equation of N for the cubic (1) is

$$u_2^2 - 9u_0u_1 = 0. \tag{3}$$

* *Higher Plane Curves*, third edition, Art. 149 ff.

† "A Memoir on Curves of the Third Order," *Phil. Trans.* (1857), p. 439, or *Collected Papers*, Vol. II, p. 405.

‡ *Phil. Trans.* (1888) A, p. 170 and *Proc. Lond. Math. Soc.*, Vol. XXI (1890), p. 247.

§ University lectures for the session 1910-1911.

|| Morley, American Journal of Mathematics, Vol. XI (1889), p. 316; Winger, "Involutions on the Rational Cubic," *Bulletin, Amer. Math. Soc.*, October, 1918, p. 28. This conic touches the nodal tangents where they meet the line of flexes and has contacts with the curve at the sextactic points.

While every line u' has a unique satellite u, a line u is in general satellite to four lines u'. Continuing the figure we shall call the line u' the *primary* of u. Combining these statements we have the theorem: *The 6 contacts of tangents from 3 collinear points (on u) of a rational cubic lie by threes on the 4 sides u' of a quadrilateral whose diagonal 3-line is circumscribed to N.*

When u' is tangent to the curve, the satellite is likewise tangent, at the point where u' cuts again. If however u is tangent two of the primary lines coincide. Thus the tangent at a point is satellite to the two tangents from the point and the line, counted twice, joining contacts of those tangents.

2. The line $u' \equiv (ux) \equiv u_0x_0 + u_1x_1 + u_2x_2 = 0$

cuts the curve in three points with parameters

$$u_2t^3 + 3u_0t^2 + 3u_1t + u_2 = 0. \tag{4}$$

If s_i refer to the symmetric functions of these t's and σ_i to the parameters $(-1/t^2)$ of the three tangential points we have

$$s_1 = -3u_0/u_2, \quad s_2 = 3u_1/u_2, \quad s_3 = -1, \tag{5}$$
$$\sigma = -\Sigma\, 1/t_i^2 = -(2s_1 + s_2^2), \quad \sigma_2 = s_1^2 - 2s_2, \quad \sigma_3 = -1.$$

Whence the equation giving the tangential points is

$$t^3 + (2s_1 + s_2^2)t^2 + (s_1^2 - 2s_2)t + 1 = 0 \tag{6}$$

and the equation of the line u on which they lie is

$$\Sigma_2:\ (3u_1^2 - 2u_0u_2)x_0 + (3u_0^2 - 2u_1u_2)x_1 + u_2^2x_2 = 0. \tag{7}$$

For given u' (7) is the equation of the satellite line. Equally for given x and variable u it is the equation of a conic, the locus of lines whose satellites pass through x. When considered as a conic we shall denote (7) by Σ_2.

If x is on the curve the conic degenerates to the two contacts of tangents from x. Hence the discriminant of the conic is to within a factor the point equation of the R_4^3.

3. If x is not on the cubic 4 tangents u_i can be drawn to the curve. From each point of contact t_i canbe drawn 2 tangents u'_{i_1} and u'_{i_2}, with contacts t'_{i_1} and t'_{i_2}. These tangents u' and the line u'_{in} joining their contacts are all primaries of u_i. *Hence the four pairs of tangents u'_{i_1}, u'_{i_2} which can be drawn to the R_4^3 from the contacts t_i of tangents from a point x, together with the four lines u'_{in} joining contacts t'_{i_1}, t'_{i_2} of the pairs touch a conic Σ_2. The four pairs of tangents are the common lines of conic and R_4^3 and the four lines u'_{in} are the common lines of Σ_2 and conic N.*

Continuing, from each of the 8 points t' can be drawn a pair of tangents u'' with contacts t'' and these contacts determine a line u''_n. *These* 24 *lines all touch a curve of class four, viz. the locus of lines whose satellites envelop the conic* Σ_2. *The* 16 *lines* u'' *are the common lines of the quartic and the* R_4^3 *while the* 8 *lines* u''_n *are the common lines of the quartic and the conic* N.

Starting with a curve Σ_1 of class one, namely the point x, we have thus a chain of theorems on class curves associated with the rational cubic for the process can be continued indefinitely. This chain is however a special case of another which can be obtained by direct generalization of the theorem implied by equation (7). Since the equation of the satellite line is of the second degree in the coefficients of the primary, we may say: *If the satellite line* u *run around a curve* Σ_δ *of class* δ, *the primary* u' *envelops a curve* $\Sigma_{2\delta}$ *of class* 2δ. *The equation of* $\Sigma_{2\delta}$ *can be found at once by replacing* u_0, u_1, u_2 *in the homogeneous line equation of* Σ_δ *by*

$$(3u_1^2 - 2u_0u_2) : (3u_0^2 - 2u_1u_2) : u_2^2$$

respectively.

Now the R_4^3 and Σ_δ have 4δ common lines u. If the contacts be designated by t, then from each t can be drawn two tangents u' to the cubic. The contacts t' of these tangents determine a line u'_n. These lines u' are the three primaries of u and hence are lines of $\Sigma_{2\delta}$. *We have thus* 12δ lines of $\Sigma_{2\delta}$,—*the* 8δ *lines* u' *which are the common lines of* $\Sigma_{2\delta}$ *and* R_4^3 *and the* 4δ *lines* u'_n *which the common lines of* $\Sigma_{2\delta}$ *and the conic* N. From the 8δ points t' can be drawn in turn 8δ pairs of tangents u'' which determine 8δ lines u''_n joining contacts t'' of the pairs. *These lines all touch a curve* $\Sigma_{4\delta}$ *of class* 4δ, *the locus of lines whose satellites envelop* $\Sigma_{2\delta}$. *The* 16δ *lines* u'' *are the common lines of* $\Sigma_{4\delta}$ *and the cubic and the* 8δ *lines* u''_n *are the common lines of* $\Sigma_{4\delta}$ *and the conic* N. *And so on ad infinitum.* Hence any curve Σ_δ establishes a chain which reduces to the old when $\delta = 1$.

4. We come now to the converse problem: given a class curve to find the satellite locus. Theorem. *If a line touch a curve* S_δ *of class* δ, *the satellite envelops a curve* $S_{2\delta}$ *of class* 2δ.* *In particular if* S_δ *is rational so also is* $S_{2\delta}$. If we denote the satellite locus by S_μ, then we want the number of lines of S_μ which pass through an arbitrary point. Now those lines u' of S_δ whose satellites pass through a given x touch a conic Σ_2. But S_δ and the conic have just 2δ lines in common i. e. there are 2δ lines of S_μ on x and $\mu = 2\delta$.

* This involves no contradiction with the theorem of the previous section; for each line u has four primaries u', hence as u' runs around $\Sigma_{2\delta}$, u will generate an $S_{4\delta}$, which is Σ_δ repeated four times.

To prove the second part of the theorem, let

$$u_0(t)x_0+u_1(t)x_1+u_2(t)x_2=0, \tag{8}$$

where the u's are binary forms of order δ be the map equation in lines of a rational curve R_δ of class δ. The map equation of the satellite locus is found by replacing the coefficients of the x's by

$$(3u_1^2-2u_0u_2) : (3u_0^2-2u_1u_2) : u_2^2$$

respectively. The result is obviously an equation in which the coefficients of the x's are rational and of degree 2δ in t.

To find the common lines of $S_{2\delta}$ and R_4^3 it is only necessary to recall that the satellite is tangent to the cubic only if the primary is a tangent of the cubic or a line of conic N. S_δ and R_4^3 have 4δ common lines whose satellites, viz. the tangents at the points in which the lines cut R_4^3 again, account for 4δ common lines of R_4^3 and $S_{2\delta}$. The 2δ primaries which touch the conic N therefore must take up the remaining 4δ common lines, i. e. each of these satellites will count for two.

As an application of these theorems we observe

(*a*) *The satellite locus of the conic N is the cubic itself.*

(*b*) *The cubic taken twice is its own satellite locus.*

(*c*) *The locus of lines whose satellites touch the cubic is a composite curve of class eight consisting of the cubic and the conic N repeated.*

THE NON-SINGULAR CUBIC.

5. Most of the theorems stated for the rational cubic can be extended at once to the case of the general cubic C_6^3. The tangent at a point P is now satellite to 10 lines, viz. the 4 tangents from the point and the 6 lines (repeated) joining in pairs the contacts of these tangents. Salmon * has remarked that the envelope of the 6 lines for variable P is a certain composite curve M_9 consisting of three class cubics.

Morley † has shown that the satellite of the line (ux) is

$$\{\Omega^2-\tfrac{1}{6}(ux)\Omega^3\}\, C^3\Gamma_6 \tag{9}$$

where C^3 and Γ_6 are respectively the point and line equations of the curve and Ω is the ternary differential operator

$$\left(\frac{\partial^2}{\partial x_0\partial u_0}+\frac{\partial^2}{\partial x_1\partial u_1}+\frac{\partial^2}{\partial x_2\partial u_2}\right)$$

* *Higher Plane Curves*, Art. 151.

† Loc. cit.

Calculated for the canonical form

$$x_0^3 + x_1^3 + x_2^3 + 6ax_0x_1x_2 = 0, \tag{10}$$

the equation of the satellite line is

$$\Gamma_4 : \sum_0^2 \{u_i^4 - 2(u_j^3 + u_k^3)u_i - 6a\, u_j^2 u_k^2\} x_i = 0, \quad i \neq j \neq k \tag{11}$$

Again for given x equation (11) represents a class quartic Γ_4, the locus of lines whose satellites are on x. This curve is of the Humbert or *desmic* type.* A line quartic is desmic if it belongs to a pencil of quartics which contains three degenerate curves each composed of four points, the three degenerate curves then being desmic quadrangles. That Γ_4 is desmic can be shown as follows. Consider a line u cutting the cubic in points a, b and c. Denote the contacts of tangents from these points by α_i, β_i, γ_i, $(i = 1, 2, 3, 4)$. Any point x on u will determine a curve Γ_4. Two points x and x' thus determine a pencil

$$\Gamma_4 + \lambda \Gamma'_4 \tag{12}$$

of curves associated with the points $x + \lambda x'$ of u. Now the primary lines of the (line) pencil on a are obviously the four (line) pencils on points α_i. In other words points α constitute a degenerate member of the pencil (12). This pencil of quartics thus contains the three desmic quadrangles α, β, γ which proves the theorem.

We have the following theorems for the general cubic, omitting the proofs which follow closely those given at length for the earlier case.

If the satellite line run around a curve Γ_δ of class δ, the primary envelops a curve $\Gamma_{4\delta}$ of class 4δ whose equation is obtained by replacing u_0, u_1, u_2 in the homogenous equation of Γ_δ by the coefficients of the x's in equation (11). Γ_δ and C_6^3 have 6δ common lines u with contacts s. From each point s can be drawn to the cubic 4 tangents u' with contacts s'. Joining the points s' in pairs are 6 lines u'_m. There are thus 60δ tangents of $\Gamma_{4\delta}$ which comprise the primaries of lines u,—the 24δ tangents u' which are the common lines of $\Gamma_{4\delta}$ and the cubic, and the 36δ lines u'_m which are the common lines of $\Gamma_{4\delta}$ and the curve M_9.

This is the first link in a chain of theorems associated with every curve Γ_δ. A chain of especial interest is that originating with a point Γ_1:

* After Humbert who discusses such curves in two papers, *Journal de Mathématiques*, 4e Série, Vol. VI (1890), p. 423 and VII (1891), p. 353. Professor Morley called my attention to the fact that Γ_4 is desmic. Since it depends on eleven constants it would appear to be the general desmic quartic.

From a point x can be drawn in general 6 tangents u with contacts s. From each point s can be drawn 4 tangents u' with contacts s' and the points s' can be joined in pairs by 6 lines u'_m. The 24 tangents u' and 36 lines u'_m touch a curve Γ_4 of class four, namely the locus of lines whose satellites pass through x. The tangents u' are the common lines of Γ_4 and C_6^3 while the lines u'_m are the common lines of Γ_4 and M_9. Again the 24 points s' determine 96 tangents u'' with contacts s'' and 144 lines u''_m joining these contacts in pairs. These 240 lines u'' touch a curve Γ_{16}, of class 16, the primary locus of Γ_4, and comprise the lines which Γ_{16} has in common with C_6^3 and M_9. And so on forever.

Most of the peculiarities of Humbert's curve as summarized (for the dual) in section 12 of his first paper can be recovered readily with our present apparatus. Indeed it seems preferable to reverse his procedure and derive the properties of the quartic from the cubic and point which define it.

The six characteristic 4-points whose diagonal points are nodes and whose connecting lines are the nodal tangents are the six sets of points s' in the theorem just stated. In other words, *the 36 intersections of Γ_4 and the cubic are at these 18 nodes and the common lines of Γ_4 and M_9 are the 36 nodal tangents,—each component of M_9 touching the tangent pair of one node in each set.* Or again *from the tangentials of the points s can be drawn 18 tangents (in addition to the lines u), the contacts of which are at the 18 nodes.*

Likewise the satellite locus of a curve C_δ of class δ is a curve $C_{4\delta}$ of class 4δ rational if C_δ is rational. The common lines of $C_{4\delta}$ and C_6^3 are (a) the satellites of the 6δ common lines of C_δ and the cubic and (b) the satellites of the 9δ common lines of C_δ and M_9, the latter lines each counting for two.

The satellite locus of M_9 is the cubic repeated six times.

The satellite locus of the cubic is the curve itself four times repeated; while the primary locus consists of the cubic and M_9 repeated.

Since M_9 is the double curve in the primary locus of the cubic it must be the Jacobian of the coefficients in the satellite line.* Calculated thus and verified for special lines the equation of M_9 is found to be

$$U^3 - 36a^2U^2 - 54aU - 54(1 + 4a^3) = 0 \tag{13}$$

where $$U = \frac{u_0^3 + u_1^3 + u_2^3}{u_0u_1u_2}.$$

Hence the factors of M_9 are

$$u_0^3 + u_1^3 + u_2^3 - k_i u_0u_1u_2 = 0, \tag{14}$$

* Likewise the Jacobian of the coefficients in the satellite (7) of the rational cubic is Nu, which incidentally proves the defining property of N.

where k_i are the roots of (13) considered as a cubic in U. It follows that the three components belong to the syzygetic pencil of class cubics determined by the nine harmonic polars.

M_9 also touches the cubic at the 27 sextactic points. Hence the tangents to the cubic from the flexes account for all the common lines of the two curves.

THE GENERAL PLANE CURVE.

6. Consider now the plane curve C_δ^n of order n and class δ. The tangents at n points of a line (ux) constitute a curve T^n which meets C^n in n^2 points. Since $2n$ of these points are the complete intersections of C^n and a curve of order 2,—the line repeated,—it follows from the theory of residuation that the remaining $n(n-2)$ lie on a C^{n-2}, the satellite $(n-2)$-ic of the line u.

The equation of the satellite curve, $f(u^k x^{n-2})=0$, will contain u as well as x. For a given x therefore, $f=0$ represents a curve of class k, the locus of lines whose satellites pass through x. To ascertain the value of k, it will be sufficient to enumerate the common lines of C_μ^n and f considered as a class curve. Let u' be such a line. Then since u' is a tangent to C^n its satellite curve degenerates into the $n-2$ tangents at the remaining intersections of u' and C^n. One of these $n-2$ lines must pass through x in virtue of the defining property of f, i. e. u' is a tangent to C^n from a contact of one of the tangents from x. Since C_δ^n is of class δ, there are precisely $\delta(\delta-2)$ lines u'. Hence $k=\delta-2$.*

We have at once the following generalization of the first link of the special chain theorem for the cubic. *Denoting by $t_1, t_2, \cdots\cdots t_\delta$ the contacts of the δ tangents from an arbitrary point x to a curve C_δ^n of order n and class δ, the δ sets of $(\delta-2)$ tangents from points t_i comprise the common lines of C_δ^n and a curve f of class $(\delta-2)$, namely the locus of lines whose satellite $(n-2)$-ics pass through x.* If C_δ^n is non-singular f is of class $(n+1)(n-2)$ and there are $(n+1)(n)(n-1)(n-2)$ common lines.

UNIVERSITY OF OREGON,
January, 1918.

---

* It can be shown by the analytic method employed by Morley, l. c. that f involves the coefficients of C_δ^n to the degree $2n-1$. For an application of that method when $n=4$, see a paper on the satellite conic of the quartic by T. Cohen, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII (1916), p. 325.

THE FAILURE OF THE CLIFFORD CHAIN.

By Walter B. Carver.

The Clifford chain theorem* defines, for a set of n lines in a plane no two of which are parallel, a Clifford circle when n is odd and a Clifford point when n is even. The Clifford point for two lines is their point of intersection, and the Clifford circle for three lines is the circum-circle. For any odd n, each set of $n - 1$ lines out of the n lines determines a Clifford point, and the n such Clifford points lie on a circle, the Clifford circle of the n lines. For any even n (greater than 2), each set of $n - 1$ lines determines a Clifford circle, and the n such Clifford circles pass through a point, the Clifford point of the n lines. In his proof of the theorem, Clifford does not raise the question of the existence of exceptional sets of lines for which the theorem may fail.

Kantor pointed out† that when five lines touch a deltoid (a hypocycloid of three cusps) the five Clifford points of the sets of four lines lie on a straight line instead of a circle; and that for six lines touching a deltoid, the six Clifford *lines* for the sets of five lines are not concurrent but are tangents of a new deltoid.

In his paper "On the Metric Geometry of the Plane N-Line,"‡ Morley gives, incidentally, an analytic proof of Clifford's theorem; shows that any odd number of lines (greater than three) determine, when they satisfy a certain analytic condition, a Clifford *line* rather than a circle; and intimates that further degeneracy may occur.

It is the purpose of the present paper to consider all the possibilities of failure of the Clifford chain, and to examine the conditions on a set of lines which cause such failure.

§ 1. *Characteristic Constants; Map and Envelope Equations.*

The analysis used will be the circular coördinates and characteristic constants of Morley's paper. It will be necessary to give a few preliminary definitions, and to state certain fundamental relations to be used later. These relations will be stated without proof, as some of them may be found

* Clifford, "Synthetic Proof of Miquel's Theorem," *Messenger of Mathematics*, Vol. 5, page 124; 1870.

† Kantor, "Die Tangentengeometrie an der Steiner'schen Hypocycloid," *Wiener Sitzungsberichte*, Vol. 78, page 204; 1878.

‡ *Transactions of the Amer. Math. Soc.*, Vol. 1, page 97; 1900.

in Morley's paper and the others may be verified by the reader without difficulty.

The circular coördinates x and y of a point are defined by the equations $x = X + iY$, $y = X - iY$, where X, Y are rectangular Cartesian coördinates and i is the imaginary unit. A complex number whose absolute value is unity will be called a turn. A real line is represented by a linear equation $tx + y = ct$, where c is any complex number and t a turn such that $\bar{c} = ct$. The point* c is the reflexion of the origin in the line (briefly, the reflex point of the line), and the turn t gives the inclination of the line and will be called the clinant. Two lines with clinants t_1 and t_2 are parallel if $t_1 = t_2$, and perpendicular if $t_1 = -t_2$.

A set of n lines (no two of which are parallel) determine uniquely a set of n characteristic constants, defined as follows:

$$a_i = \frac{1}{T}\begin{vmatrix} c_1 t_1^{n-i} & t_1^{n-2} & t_1^{n-3} & \cdots & t_1 & 1 \\ c_2 t_2^{n-i} & t_2^{n-2} & t_2^{n-3} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ c_n t_n^{n-i} & t_n^{n-2} & t_n^{n-3} & \cdots & t_n & 1 \end{vmatrix}, \qquad i = 1, 2, \cdots, n,$$

where

$$T = \begin{vmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & t_1 & 1 \\ t_2^{n-1} & t_2^{n-2} & \cdots & t_2 & 1 \\ \cdots & & & & \\ t_n^{n-1} & t_n^{n-2} & \cdots & t_n & 1 \end{vmatrix}.$$

Between these n a's we have the relations

$$\bar{a}_i = (-1)^{n-1} S_n a_{n-i+1}, \qquad i = 1, 2, \cdots, n,$$

where $S_n = t_1 t_2 t_3 \cdots t_n$. Since $S_n \neq 0$, a_i and a_{n-i+1} (called complementary a's) vanish together.

To fix an arbitrary set of n lines,

(1) for n even, $a_1, a_2, \cdots, a_{n/2}$ and the n clinants $t_1, t_2, \cdots, t_n$ (no two of them equal) may be chosen arbitrarily; and the remaining a's and the set of n lines are then uniquely fixed:

(2) for n odd, $a_1, a_2, \cdots, a_{(n+1)/2}$, and any $n-1$ of the n clinants $t_1, t_2, \cdots, t_n$ may be chosen arbitrarily; and the remaining t and a's and the set of n lines are then uniquely fixed.

If one of a set of n lines is omitted, the constants for the remaining set of $n-1$ lines are

$$\alpha_i = a_i - t a_{i+1}, \qquad i = 1, 2, \cdots, n-1,$$

* The phrase "the point c" is used in the sense of the usual relation between the complex numbers and the points of a plane. The point c is the *real* point whose circular coördinates are $(c, \bar{c})$.

where t is the clinant of the omitted line. More generally, if r lines are omitted, the constants for the $n - r$ lines are

$$\alpha_i = a_i - S_1 a_{i+1} + S_2 a_{i+2} - \cdots + (-1)^r S_r a_{i+r}, \qquad i = 1, 2, \cdots, n - r,$$

where the S's are the ordinary symmetric functions* of the r clinants of the omitted lines.

A set of n lines are concurrent if, and only if,

$$a_2 = a_3 = \cdots = a_{n-1} = 0;$$

and the lines will then meet at the point a_1. If $n - r$ lines out of a set of n lines are concurrent, we must then have

$$\begin{aligned} a_2 - S_1 a_3 + S_2 a_4 - \cdots + (-1)^r S_r a_{r+2} &= 0, \\ a_3 - S_1 a_4 + S_2 a_5 - \cdots + (-1)^r S_r a_{r+3} &= 0, \\ &\vdots \\ a_{n-r-1} - S_1 a_{n-r} \quad - \cdots + (-1)^r S_r a_{n-1} &= 0, \end{aligned}$$

and the $n - r$ lines meet at the point

$$a_1 - S_1 a_2 + \cdots + (-1)^r S_r a_{r+1};$$

where the S's are for the clinants of the omitted lines. If $n - r \leqq (n/2) + 1$, this imposes no condition on the a's alone; but if $n - r > (n/2) + 1$, the matrix

$$\left\| \begin{matrix} a_2 & a_3 & \cdots & a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-r-1} & a_{n-r} & \cdots & a_{n-1} \end{matrix} \right\|$$

must be of rank r if $n - r$ (and no more) of the n lines are concurrent.

In the following sections an important rôle will be played by determinants of the form

$$\begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+k-1} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2} \end{vmatrix}$$

where the elements are the characteristic constants of a set of n lines. Such a determinant of the kth order is equal to a determinant of the nth order in the reflex points and clinants of the n lines, as follows:

$$\begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+k-1} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2} \end{vmatrix}$$

* Throughout the paper S_i, $i = 1, 2, \cdots k$, will represent the ordinary symmetric functions of k turns $t_1, t_2, \cdots t_k$; *i.e.*, the S's will be the coefficients of the equation $t^k - S_1 t^{k-1} + S_2 t^{k-2} - S_3 t^{k-3} + \cdots + (-1)^k S_k = 0$ whose roots are the k turns.

$$= \frac{(-1)^{k(k-1)/2}}{T} \begin{vmatrix} c_1 t_1^{n-i-k+1} & c_1 t_1^{n-i-k} & \cdots & c_1 t_1^{n-i-2k+2} & t_1^{n-k-1} & t_1^{n-k-2} & \cdots & t_1 & 1 \\ c_2 t_2^{n-i-k+1} & c_2 t_2^{n-i-k} & \cdots & c_2 t_2^{n-i-2k+2} & t_2^{n-k-1} & t_2^{n-k-2} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n t_n^{n-i-k+1} & c_n t_n^{n-i-k} & \cdots & c_n t_n^{n-1-2k+2} & t_n^{n-k-1} & & \cdots & t_n & 1 \end{vmatrix}$$

where

$$T = \begin{vmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & t_1 & 1 \\ t_2^{n-1} & t_2^{n-2} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ t_n^{n-1} & t_n^{n-2} & \cdots & t_n & 1 \end{vmatrix}.$$

Also such a determinant in the characteristic constants (α's) for $n-1$ out of n lines may be expressed in terms of the characteristic constants (a's) for the n lines thus

$$\begin{vmatrix} \alpha_i & \alpha_{i+1} & \cdots & \alpha_{i+k-1} \\ \alpha_{i+1} & \alpha_{i+2} & \cdots & \alpha_{i+k} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{i+k-1} & \alpha_{i+k} & \cdots & \alpha_{i+2k-2} \end{vmatrix} = \begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+k-1} & a_{i+k} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+k} & a_{i+k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2} & a_{i+2k-1} \\ t^k & t^{k-1} & \cdots & t & 1 \end{vmatrix}$$

where t is the clinant of the omitted line.

The determinants*

$$\begin{vmatrix} a_i & \cdots & a_{i+k-1} \\ \cdot & \cdot & \cdot \\ a_{i+k-1} & \cdots & a_{i+2k-2} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{n-i-2k+3} & \cdots & a_{n-i-k+2} \\ \cdot & \cdot & \cdot \\ a_{n-i-k+2} & \cdots & a_{n-i+1} \end{vmatrix}$$

will be called complementary. Two such complementary determinants must obviously vanish together.

The rational curves which occur in the paper will be represented either by "map" equations or by "envelope" equations. The map equation

$$x = R(t)$$

expresses x as a rational function of a turn parameter t. This equation always implies the corresponding equation

$$y = \bar{R}(1/t),$$

where $\bar{R}$ means the function obtained by replacing the coefficients in the function R by their conjugates. When t runs through turn values, x gives the *real* points of the curve and x and y are conjugates. If t is given values other than turn values, x and y will not (in general) be conjugates, but will be the circular coördinates of *imaginary* points of the curve.

---

* Determinants of this type will be sufficiently indicated by giving only the four corner elements.

The following special map equations may be noted:

$$x = c + kt$$

is a circle with center at c and radius $|k|$.

$$x = c + \frac{k}{(1 + kt/\bar{k})^2}$$

is a parabola of which the point c is the focus and k represents a vector from the focus to its reflexion in the directrix. In this form the parameter t is the clinant of the tangent to the curve at the point x.

$$x = c + k\left(\frac{\bar{k}^2}{k^2t^2} - \frac{2kt}{\bar{k}}\right)$$

is a deltoid of which c is the center and $3k$ represents a vector from the center to one of the cusps. In this case also t is the clinant of the tangent at the point x.

The envelope equation of a curve is of the form

$$tx + y = R(t),$$

where R is a rational function satisfying the identical relation

$$\bar{R}(1/t) = R(t)/t.$$

For a given turn value of t the envelope equation represents a straight line, and as t runs through turn values this line envelopes the curve. The map equation of the curve is obtained at once from this envelope equation by partial differentiation with respect to t, giving

$$x = R'(t),$$

where the parameter t is the clinant of the tangent at the point x.

The envelope equation of a parabola is

$$tx + y = (\alpha t^2 + rt + \bar{\alpha})/(\gamma t + \bar{\gamma}),$$

where r is real, $\gamma \neq 0$, and the denominator is not a factor of the numerator. Similarly, the envelope equation of a deltoid is

$$tx + y = (\alpha_1 t^3 + \alpha_2 t^2 + \bar{\alpha}_2 t + \bar{\alpha}_1)/rt,$$

where r is real and $\neq 0$, and $\alpha_1 \neq 0$. Differentiating these equations with respect to t, and identifying the results with the map equations of these same curves, we see that the focus of the parabola is at the point α/γ; while the center of the deltoid is at α_2/r, and its size is given by $|\alpha_1|/r$.

The most general curve which can be represented by an envelope equation is treated in sections 3 and 8.

§ 2. *Improper Sets of 3, 4, 5, and 6 Lines.*

The map equation of the Clifford circle as given by Morley is, for 3 lines,

$$x = a_1 - a_2 t,$$

and for $2p - 1$ lines, where $p \geqq 3$,

$$x = \frac{\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-3} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix}} - \frac{\begin{vmatrix} a_2 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-2} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix}} t$$

and the Clifford point for $2p$ lines, $p \geqq 2$, is

$$x = \frac{\begin{vmatrix} a_1 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}}.$$

The expression for the circle fails to represent a circle when the numerator of the fraction representing the radius vanishes while the denominator does not,* and in all cases where the denominator vanishes; and the expression for the Clifford point similarly fails whenever the denominator vanishes. Sets of lines for which these expressions thus fail will be called improper, all other sets proper. A necessary and sufficient condition that a set of lines be improper is given in theorem 13, § 5. The facts for 3, 4, 5, and 6 lines are well known; but a brief consideration of these cases is necessary to indicate the method of treatment of the general case and to establish a basis for an inductive proof.

For 3 lines the expression represents a circle except when $a_2 = 0$; and this is the necessary and sufficient condition that the three lines be concurrent. They meet at the point a_1.

For 4 lines the Clifford point is

$$x_1 = \frac{\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}}{a_3},$$

and this expression becomes meaningless when $a_3 = 0$. Since for 4 lines

* This case where the radius is zero is included among the cases of failure, first, because the Clifford circle is only defined by the Clifford points which lie on it, and if these points coincide they do not define a unique circle; and secondly, because of the close relationship of this case to the other cases of failure.

a_2 and a_3 are complementary, we have also $a_2 = 0$; so that both the numerator and denominator of the fraction vanish. This occurs when, and only when, the 4 lines are concurrent, meeting at the point a_1.

For 5 lines the Clifford circle is

$$x = \frac{\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}}{a_3} - \frac{\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}}{a_3} t$$

and failure occurs when

$$(1) \quad \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0, \qquad a_3 \neq 0;$$

$$(2) \quad a_3 = 0, \qquad \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0, \qquad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0;$$

$$(3) \quad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_3 = 0.$$

Case (1), $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$, $a_3 \neq 0$. Since $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$, we must have

$$\begin{vmatrix} c_1t_1^2 & c_1t_1 & t_1^2 & t_1 & 1 \\ c_2t_2^2 & c_2t_2 & t_2^2 & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_5t_5^2 & c_5t_5 & t_5^2 & t_5 & 1 \end{vmatrix} = 0.$$

Not all of the minors of the elements in the first column can vanish. For these minors are the a_3's for the sets of 4 out of the 5 lines; and if they all vanished we would have $a_3 - a_4t = 0$ for five different values of t, and hence it would follow that $a_3 = 0$, contrary to hypothesis. Suppose the non-vanishing minor to be in the upper right-hand corner. Then the reflex point c and clinant t of each of the 5 lines must satisfy the equation

$$\text{(I)} \quad A_1ct^2 + A_2ct + B_1t^2 + B_2t + B_3 = 0,$$

where the A's and B's are the co-factors of the elements of the lower row, and $A_1 \neq 0$. Solving for ct we have

$$ct = (-B_1t^2 - B_2t - B_3)/(A_1t + A_2);$$

and multiplying both numerator and denominator by $1/t_1t_2t_3t_4$, this takes the form

$$\text{(II)} \quad ct = (\alpha t^2 + rt + \bar{\alpha})/(\gamma t + \bar{\gamma}),$$

where r is real and $\gamma \neq 0$. Since the c and t for each of the 5 lines satisfy (I), they must also satisfy (II) unless the value of t is such as to cause the denominator $\gamma t + \bar{\gamma}$ to vanish. If this is true for any t, this t must also

cause the numerator to vanish; and in this case the denominator would be a factor of the numerator. If then in equation (II) the denominator is not a factor of the numerator, this equation is satisfied by all of the 5 lines; and the 5 lines are evidently tangents of the parabola whose envelope equation is

$$tx + y = (\alpha t^2 + rt + \overline{\alpha})/z\gamma t + \overline{\gamma}).$$

If, on the other hand, $\gamma t + \overline{\gamma}$ is factor of the numerator, we may write

$$\text{(III)} \quad ct = \alpha' t + \overline{\alpha}',$$

and this equation must be satisfied by each of the 5 lines except possibly one whose clinant makes $\gamma t + \overline{\gamma}$ vanish. But all lines satisfying (III) are concurrent, and not all of our 5 lines can be concurrent since $a_3 \neq 0$. Hence 4, and only 4, of the 5 lines are concurrent.

If, then, $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$ and $a_3 \neq 0$, either the 5 lines touch a parabola or just 4 of them are concurrent; and, conversely, if 5 lines touch a parabola or just 4 of them are concurrent, then $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$ and $a_3 \neq 0$.

Case (2), $a_3 = 0$, $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0$, $\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0$. Obviously $a_2 \neq 0$ and $a_4 \neq 0$; and $a_3 = 0$ gives

$$\begin{vmatrix} c_1 t_1^2 & t_1^3 & t_1^2 & t_1 & 1 \\ c_2 t_2^2 & t_2^3 & t_2^2 & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_5 t_5^2 & t_5^3 & t_5^2 & t_5 & 1 \end{vmatrix} = 0.$$

Hence each of the 5 lines satisfies the equation

$$Act^2 + B_1 t^3 + B_2 t^2 + B_3 t + B_4 = 0,$$

where the A and the B's are co-factors of the elements of the lower row, and $A \neq 0$. Dividing through by $1/(t_1 t_2 t_3 t_4)^{3/2}$, and solving for ct, we have

$$ct = (\alpha_1 t^3 + \alpha_2 t^2 + \overline{\alpha}_2 t + \overline{\alpha}_1)/(rt),$$

where r is real and $\neq 0$. Since $t \neq 0$ for any line, all of the 5 lines must satisfy this equation. Moreover, we can not have $\alpha_1 = 0$; for this would give $ct = \alpha' t + \overline{\alpha}'$, which would make the 5 lines concurrent, contrary to the condition $a_2 \neq 0$. Hence all of the 5 lines must touch the deltoid whose envelope equation is

$$tx + y = (\alpha_1 t^3 + \alpha_2 t^2 + \overline{\alpha}_2 t + \overline{\alpha}_1)/(rt).$$

Conversely, if 5 lines touch a deltoid we must have $a_3 = 0$ and

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0, \qquad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0.$$

Case (3), $\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_3 = 0.$ It follows at once that $a_2 = a_4 = 0$, and hence that the 5 lines are concurrent at the point a_1.

For 6 lines the Clifford point is

$$x_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \Bigg/ \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}$$

and failure occurs when

$$(1) \quad \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_4 & a_5 & a \end{vmatrix} \neq 0;$$

$$(2) \quad \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 0.$$

Case (1), $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0,$ $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \neq 0.$ With $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0$ we have the complementary condition $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0.$ Then unless $a_3 = a_4 = 0$, all the determinants of the 2nd order in the matrix $\begin{Vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{Vmatrix}$ would vanish, causing $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$ to vanish. It follows that $a_3 = a_4 = 0$, and therefore also that $a_2 \neq 0$, $a_5 \neq 0$. Since $a_3 - a_4 t = 0$ for all values of t, we have $\alpha_3 = 0$ for every set of 5 put of the 6 lines. Moreover, $\begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix}$ can not vanish for any set of 5; for if it did, it would follow for that set of 5 lines that $\alpha_2 = \alpha_3 = \alpha_4 = 0$ or, in terms of the a's for the 6 lines, $a_2 - a_3 t = a_3 - a_4 t = a_4 - a_5 t = 0$, contrary to the condition, $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \neq 0.$ Hence each set of 5 out of the 6 lines touch a deltoid; and since a deltoid is uniquely determined by four of its tangents, it follows that the 6 lines all touch one deltoid. And conversely, if 6 lines touch a deltoid, we must have $a_3 = a_4 = 0$, and $a_2 \neq 0$, $a_5 \neq 0$.

Case (2), $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 0.$ As in case (1), $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$

also. If $a_3 = 0$, it follows that $a_4 = a_2 = a_5 = 0$, and hence that the 6 lines are concurrent. If on the contrary $a_3 \neq 0$, then $a_4 \neq 0$, $a_2 \neq 0$, $a_5 \neq 0$, and all the determinants of the 2nd order in the matrix $\begin{Vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{Vmatrix}$ vanish. Since then $\begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ t^2 & t & 1 \end{vmatrix} = 0$ for all values of t, we have $\begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix} = 0$ for every set of 5 out of the 6 lines. Also α_3 can not vanish for more than one set of 5. There are then two possibilities in this case, viz., either all of the 6 lines touch a parabola, or just 5 of them are concurrent. And conversely, if 6 lines touch a deltoid, or if just 5 of them are concurrent, then all the determinants of 2nd order in the matrix $\begin{Vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{Vmatrix}$ vanish, and $a_2 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$, $a_5 \neq 0$.

§ 3. *The Curves $P_k^{(n)}$.*

A polynomial of the form

$$\alpha_1 t^n + \alpha_2 t^{n-1} + \cdots + \bar{\alpha}_2 t + \bar{\alpha}_1 \equiv P(t),$$

in which the coefficients of t^i and t^{n-i} are conjugates, may be called a *self-conjugate polynomial of degree n in t.* If n is even, the coefficient of $t^{n/2}$ is real. Such a polynomial has a theory analogous to that of a polynomial with real coefficients. Expressing it as a product of linear factors,

$$P(t) \equiv \alpha_1(t - \rho_1)(t - \rho_2) \cdots (t - \rho_n),$$

all of the ρ's which are not turns must pair off into inverse pairs, *i.e.*, into pairs ρ_1 and ρ_2 such that $\rho_1\bar{\rho}_2 = \bar{\rho}_1\rho_2 = 1$. Thus the number of non-turn roots of $P(t) = 0$ is always even; and if n is odd, at least one root must be a turn. Any such self-conjugate polynomial may be decomposed into self-conjugate polynomial factors, no factor being of degree higher than the second. If a self-conjugate polynomial is divisible by another self-conjugate polynomial, the quotient is also self-conjugate; and the highest common factor of two self-conjugate polynomials is a self-conjugate polynomial. It will be convenient to consider

$$\alpha_{i+1} t^{n-i} + \alpha_{i+2} t^{n-i-1} + \cdots + \bar{\alpha}_{i+2} t^{i+1} + \bar{\alpha}_{i+1} t^i$$

as a self-conjugate polynomial *of degree n.* It may be understood that 0 and ∞ are i-fold inverse roots in this case. Thus $t^3 + t^2$ is a self-conjugate polynomial of degree 5 rather than of degree 3. In breaking this polynomial up into self-conjugate factors, the factor t is to be regarded as a self-conjugate polynomial of degree 2.

The notation $P_k^{(m)}$ will indicate the curve whose envelope equation is

$$tx + y = \frac{\alpha_1 t^m + \alpha_2 t^{m-1} + \cdots + \bar{\alpha}_2 t + \bar{\alpha}_1}{\gamma_{k+1} t^{m-k-1} + \gamma_{k+2} t^{m-k} + \cdots + \bar{\gamma}_{k+2} t^{k+1} + \gamma_{k+1} t^k} \equiv \frac{\alpha(t)}{\gamma(t)}$$

where $m \geqq 1$; $0 \leqq k \leqq (m-1)/2$; $\gamma_{k+1} \neq 0$; and $\alpha(t)$ and $\gamma(t)$ are self-conjugate polynomials of degree m and $m-1$ respectively, with no common polynomial factor. From this last restriction it follows that for $k > 0$, $\alpha_1 \neq 0$; and it may also be understood to imply that $\alpha(t)$ can not vanish identically except in the rather trivial case $m = 1$, $k = 0$. The map equation of this curve is

$$x = \frac{d}{dt}\left[\frac{\alpha(t)}{\gamma(t)}\right] = \frac{\gamma(t)\alpha'(t) - \alpha(t)\gamma'(t)}{\gamma^2(t)}.$$

It is a rational curve of class m. Assuming for the moment the projective point of view, we may say that the curve is tangent to the line at infinity in the $m-1$ directions given by the roots of $\gamma(t) = 0$ (real directions for roots that are turns, pairs of imaginary directions for the pairs of inverse roots). If μ is the multiplicity of a root of $\gamma(t) = 0$, the curve has contact of the μth order in the corresponding direction. In particular, corresponding to the roots 0 and ∞, there is contact of the kth order at the circular points I and J. If σ represents the number of *distinct* roots of $\gamma(t) = 0$, the curve is of order $m + \sigma - 1$; the line at infinity counts as

$$\tfrac{1}{2}[(m-1)(m-4) + 2\sigma]$$

double tangents and as $m - \sigma - 1$ flex tangents, thus accounting for all the line singularities. There are $m + 2\sigma - 4$ cusps and

$$\tfrac{1}{2}[(m+\sigma)^2 - 7m - 9\sigma + 14]$$

nodes, these cusps and nodes not being necessarily distinct. Two such curves with the same values for m and k have m^2 common tangents, of which the line at infinity counts for at least* $(m-1)^2 + 2k$. Hence they have, aside from the line at infinity, not more than $2m - 2k - 1$ common tangents; and therefore such a curve is uniquely determined by $2m - 2k$ of its tangents. Two curves $P_k^{(m_1)}$ and $P_k^{(m_2)}$ have $m_1 m_2$ common tangents, of which the line at infinity counts for at least $(m_1 - 1)(m_2 - 1) + 2k$. Hence they have, aside from the line at infinity, not more than $m_1 + m_2 - 2k - 1$ common tangents.

This curve $P_k^{(m)}$ is the most general type of curve that can be represented by an envelope equation. It is the dual of the Jonquieres type,

---

* The line at infinity counts for more than $(m-1)^2 + 2k$ of the common tangents in case the equations $\gamma(t) = 0$ for the two curves have common roots other than 0 and ∞.

since it is rational and the line at infinity furnishes all of its line singularities. For $k = 0$ it is the $(m-1)$-fold parabola* of Clifford's paper. As special cases, it may be noted that a $P_0^{(1)}$ is merely a point, a $P_0^{(2)}$ is an ordinary quadratic parabola, and a $P_1^{(3)}$ is a deltoid. The curve $\Delta^{(2n-1)}$ of Morley† and Stephens‡ is a $P_{n-1}^{(2n-1)}$; and the curve $K^{(2n)}$ of Atchison§ is a $P_{n-1}^{(2n)}$.

§ 4. *Characteristic Matrices for N Lines.*

Let the positive integer p be defined by the relations $n = 2p - 1$ when n is odd and $n = 2p$ when n is even; and let $[h, i]$, where $h \geqq 1$, $i \leqq p$, $i - h \geqq 0$, represent the matrix||

$$\left\| \begin{matrix} a_h & a_{h+1} & \cdots & a_{n-i+1} \\ a_{h+1} & a_{h+2} & \cdots & a_{n-i+2} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & a_{i+1} & \cdots & a_{n-h+1} \end{matrix} \right\|.$$

For a set of n lines there is a triangular table of such characteristic matrices, as follows:

$$\begin{array}{llllllll} [1, 1] & & & & & & & \\ [1, 2] & [2, 2] & & & & & & \\ \vdots & \vdots & & & & & & \\ [1, i] & [2, i] & \cdots & [h, i] & \cdots & [i, i] & & \\ \vdots & \vdots & & \vdots & & & & \\ [1, p-1] & [2, p-1] & \cdots & [h, p-1] & \cdots & [p-1, p-1] & & \\ [1, p] & [2, p] & \cdots & [h, p] & \cdots & [p-1, p] & & [p, p]. \end{array}$$

In this table, a constant h gives a column, a constant i gives a row, a constant difference $i - h$ gives a principal diagonal (running downwards to the right), and a constant sum $i + h$ gives a skew diagonal (upwards to the right). For $n = 2p - 1$ the difference between the number of columns and the number of rows in each matrix is even, and the matrices of the lower row are square; while for $n = 2p$ this difference is odd, and there are no square matrices. All the matrices are self-complementary in the sense that two elements symmetrically situated with respect to the center are complementary.

---

* Clifford does not state explicitly that his multi-fold parabola must not be tangent to the line at infinity at the circular points, but he ignores the consequences which would result from such specialization of his curves.

† "Orthocentric Properties of the Plane N-Line," *Trans. of the Amer. Math. Soc.*, Vol. 4, page 1; 1903.

‡ "On the Pentadeltoid," *Trans. of the Amer. Math. Soc.*, Vol. 7, page 207; 1906.

§ "Curves with a Directrix," Johns Hopkins dissertation, 1908.

|| This matrix is also a function of n, but it is simpler to omit the letter n from the symbol.

We shall be concerned with the vanishing of the determinants of highest order in these matrices, and the following notation will be used:

$[h, i] = 0$ means that every determinant of highest order vanishes;

$[h, i] \neq 0$ means that at least one determinant of highest order does not vanish;

$\{[h, i]\} = 0$ means that every solid* determinant of highest order vanishes;

$\{[h, i]\} \# 0$ means that no solid determinant of highest order vanishes;

$\}[h, i]\{ = 0$ means that at least one solid determinant of highest order vanishes;

$\{[h, i]\} \doteq 0$ means that all the solid determinants of the matrix vanish except the first and the last which do not vanish.

The two following well-known theorems will be used:

THEOREM A.† If D is a determinant of order m, A_{11}, A_{1m}, A_{m1}, and A_{mm} are the first minors of the four corner elements, and K is the second minor obtained by striking off a border one element wide all around; then

$$DK = \begin{vmatrix} A_{mm} & A_{m1} \\ A_{1m} & A_{11} \end{vmatrix}.$$

THEOREM B.‡ If in a matrix a determinant of order r does not vanish, and if every determinant of order $r + k$ which contains this non-vanishing determinant vanishes; then every determinant of order $r + k$ in the matrix vanishes.

THEOREM 1. If $[h, i] = 0$, then also $[h', i'] = 0$ for any h' and i' such that $i' + h' \geqq i + h$ and $i' - h' \geqq i - h$.

These matrices $[h', i']$ cover an area in the table of matrices which is an isosceles triangle with vertex at the matrix $[h, i]$ and base in the lowest row, or such portion of such a triangle as is not cut off by the left-hand boundary of the table. The truth of the theorem is obvious from the fact that the matrix $[h, i]$ furnishes a band extending horizontally across the matrix $[h', i']$.

THEOREM 2. If $[h, i] = 0$, then $\{[h', i']\} = 0$ for any h' and i' such that $i - h \leqq i' - h' \leqq n - h - i + 1$ and $i' \geqq i$.

The matrices $[h', i']$ of this theorem make up a parallelogram with the matrix $[h, i]$ at one vertex and its base in the lowest row, or such portion of such a parallelogram as is not cut off by the left hand boundary of the table. (For $h = 1$ this theorem adds no information to that given by theorem 1.) The proof consists in the fact that the matrix $[h, i]$ furnishes

* A solid determinant of highest order means one made up of consecutive columns of the matrix.

† Bôcher, "Higher Algebra," page 33.

‡ This theorem is a slight extension of the one given by Bôcher, "Higher Algebra," page 54.

a band extending horizontally across any solid determinant of the matrix $[h', i']$.

THEOREM 3. If $\{[h, i]\} = 0$ and $i - h \geqq 1$, then either $\{[h+1, i]\} = 0$ or $\{[h+1, i]\} \neq 0$.

The condition $i - h \geqq 1$ is needed in order that the matrix $[h+1, i]$ should exist. For $i = p$, $n = 2p - 1$, the theorem is trivial. For $i = p$, $n = 2p$, it follows from the fact that the two solid determinants in the matrix $[h+1, i]$ are complementary and must vanish together. For $i < p$, Theorem A shows that any three successive solid determinants of the matrix $[h, i-1]$ are such that the square of the middle one is equal to the product of the two adjacent ones. It follows at once that either $\{[h+1, i]\} = 0$ or $\{[h+1, i]\} \neq 0$.

The notation $\{[h, i]\} \neq 0*$ will be used to indicate that either $\{[h, i]\} \neq 0$ or else $h > i$ and therefore the matrix $[h, i]$ does not exist.

THEOREM 4. If $\{[h, i]\} = 0$ and $\{[h+1, i]\} \neq 0*$, then $\{[h, i-1]\} \neq 0*$ and $[h, i] = 0$.

If $h = i$, the matrix $[h+1, i]$ does not exist, and also the matrix $[h, i-1]$ does not exist. Also, in this case, there will be only one row in the matrix $[h, i]$, and hence the statements $\{[h, i]\} = 0$ and $[h, i] = 0$ are identical.

If $i > h$, then $\{[h+1, i]\} \neq 0$, and from the argument for the preceding theorem it follows that $\{[h, i-1]\} \neq 0$. Then the statement $[h, i] = 0$ will follow from $\{[h, i]\} = 0$ and $\{[h+1, i]\} \neq 0$ by a repeated use of Theorem B.

THEOREM 5. If $[h, i] = 0$ and $\{[h+1, i]\} \neq 0*$, and

$$\}[h+i+j-n-2, j]\{ = 0$$

where $j \geqq i - 1$; then $[h-1, i-1] = 0$.

From $[h, i] = 0$ it follows that $\{[h', i']\} = 0$ over a parallelogram area as given in Theorem 2. The matrix $[h+i+j-n-2, j]$ is in the principal diagonal immediately to the left of this parallelogram. The existence of this matrix gives us the additional inequality $h + i + j - n - 2 \geqq 1$. All the solid determinants of $[h-1, i-1]$ except possibly the first and last (which are complementary), vanish because of the condition $[h, i] = 0$. To prove therefore that $\{[h-1, i-1]\} = 0$ it is only necessary to show that either the first or last solid determinant in the matrix vanishes.

Let

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{j+\lambda} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ a_{j+\lambda} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

where $0 \leqq \lambda \leqq n - 2j + 1$, be the vanishing solid determinant of the

matrix $[h+i+j-n-2,\ j]$; and consider first the case where $i=h$ and therefore $[h+1,\ i]$ does not exist. The condition $[h,\ i]=0$ in this case is simply $a_h = a_{h+1} = \cdots = a_{n-h+1} = 0$; and the vanishing determinant takes the form

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{h-1} & 0 & \cdots & 0 \\ a_{h+i+j-n-1+\lambda} & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{h-1} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & a_{n-h+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & a_{n-h+2} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

or $a_{h-1}^r a_{n-h+2}^s = 0$, where $r = n-i-j+1-\lambda$ and $s = j-i+2+\lambda$. Since a_{h-1} and a_{n-h+2} are complementary, it follows that they both vanish, and that therefore $\{[h-1,\ i-1]\}=0$; and this is the same, in this case, as $[h-1,\ i-1]=0$.

Suppose next that $i-h>0$, and that therefore $[h+1,\ i]$ exists. The vanishing solid determinant of $[h+i+j-n-2,\ j]$ may be written in the form

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{h-1} & a_h & \cdots & \cdots & \cdots & a_{j+\lambda} \\ a_{h+i+j-n-1+\lambda} & \cdots & a_h & a_{h+1} & \cdots & \cdots & \cdots & a_{j+\lambda+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{h-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{n-i+1} \\ a_h & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{n-i+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j-i+h+\lambda} & \cdots & \cdots & \cdots & \cdots & a_{n-i+1} & \cdots & \cdots \\ a_{j-i+h+\lambda+1} & \cdots & \cdots & a_{n-2i+h+3} & \cdots & a_{n-i+2} & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j+\lambda} & \cdots & \cdots & a_{n-i+2} & \cdots & a_{n-h+1} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

From the fact that $[h,\ i]$ is of rank $i-h$, it follows that there is a (homogeneously) unique set of constants $\lambda_1, \lambda_2, \cdots, \lambda_{i-h+1}$ such that if the elements of each row of this matrix $[h,\ i]$ be multiplied by the corresponding λ, the sum of each column will then be zero; and one may use for the λ's the $i-h+1$ determinants of order $i-h$ in the first $i-h$ columns. If then in the above determinant we multiply the first $i-h+1$ top rows respectively by these λ's, and add for a new top row; and do the same for each successive block of $i-h+1$ rows down to the lowest block of $i-h+1$ rows; we obtain the following vanishing determinant

$$\begin{vmatrix} \cdots & \cdots & D_1 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & D_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 & D_2 & \cdots & \cdots \\ a_{j-i+h+\lambda+1} & \cdots & \cdots & a_{n-2i+h+3} & \cdots & a_{n-i+2} & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j+\lambda} & \cdots & \cdots & a_{n-i+2} & \cdots & a_{n-h+1} & \cdots & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix} = 0;$$

or $D_1^r D_2^s D_3 = 0$, where

$$D_1 = \begin{vmatrix} h-1 & \cdots & i-1 \\ \cdots & \cdots & \cdots \\ i-1 & \cdots & 2i-h-1 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} n-i+2 & h & h+1 & \cdots & i-1 \\ \cdots & \cdots & \cdots & & \cdots \\ n-h+2 & i & i+1 & \cdots & 2i-h-1 \end{vmatrix},$$

and

$$D_3 = \begin{vmatrix} n-2i+h+3 & \cdots & n-i+2 \\ \cdots & & \cdots \\ n-i+2 & \cdots & n-h+1 \end{vmatrix}.$$

$D_3 \neq 0$; for it is the last solid determinant in $[h, i-1]$, and from Theorem 4, $\{[h, i-1]\} \# 0$. Hence either D_1 or D_2 must vanish. D_1 is the first solid determinant in $[h-1, i-1]$; and if $D_2 = 0$, by the repeated use of Theorem B, with the hypothesis $[h, i] = 0$, it will follow that the last solid determinant in $[h-1, i-1]$ vanishes. This completes the proof that $\{[h-1, i-1]\} = 0$. But with $\{[h, i-1]\} \# 0$, Theorem 4 gives $[h-1, i-1] = 0$.

THEOREM 6. If $[h, i] = 0$ and $\{[h+1, i]\} \# 0$ * and

$$[h+i+i'-1, i'] = 0,$$

where $i' \lesseqgtr i$; then $[h-1, i-1] = 0$.

The matrix $[h+i-i'-1, i']$ is any matrix in the skew diagonal immediately to the left of the matrix $[h, i]$. By Theorem 2,

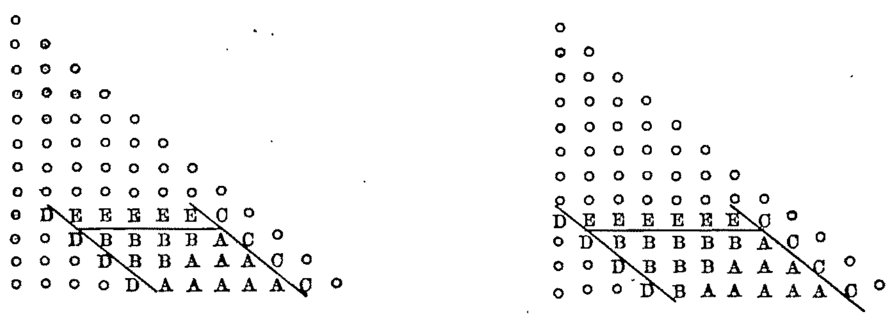
$$[h+i-i'-1, i'] = 0$$

gives $\{[h+i+i'-n-2, i']\} = 0$. This, with $[h, i] = 0$ and $\{[h+1, i]\} \# 0$ *, satisfies the hypothesis of Theorem 5, and hence gives $[h-1, i-1] = 0$.

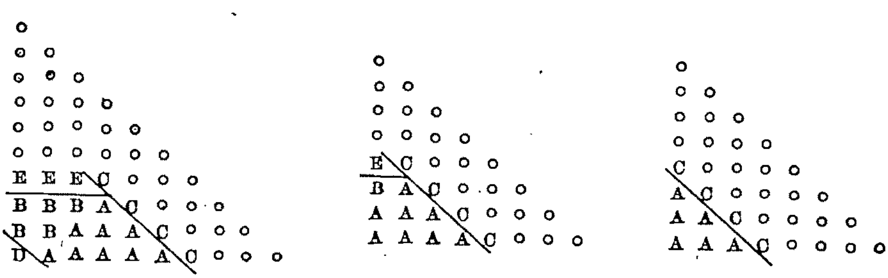
By means of the six foregoing theorems, it may be shown that any characteristic matrix for n lines which has the property $[h, i] = 0$ or

$\{[h, i]\} = 0$ lies in a "vanishing area" in the triangular table of matrices; this vanishing area being a parallelogram or portion of a parallelogram. Every such vanishing parallelogram extends downwards to include the lowest (or pth) row; and is made up of matrices of two kinds A and B, as shown in the figure below, having the respective properties $[A] = 0$ and $\{[B]\} = 0$. For n odd, the matrices in the upper right and lower left corners of the parallelogram are both A's, and lie on the same skew diagonal. For n even, the upper right matrix is an A while the lower left is a B, and the skew diagonal containing the former is immediately to the right of that containing the latter. This vanishing parallelogram is bordered on three sides by a non-vanishing border of matrices C, D, and E, having the properties respectively $\{[C]\} \# 0 *$, $\{[D]\} \# 0$, $\{[E]\} \doteq 0$.

A vanishing area may be only part of such a parallelogram, a part at the left being cut off by the left-hand boundary of the table, as shown in the following figures:



Vanishing parallelogram for n odd.



Vanishing parallelogram for n even.

Because of the non-vanishing border, two vanishing areas can obviously not overlap.

The matrix $[h, i]$, in which $h \geqq 2$, will be called a *key matrix*

(a) when $h = 2$, if $[h, i] = 0$ and $\{[h + 1, i]\} \# 0 *$; and

(b) when $h > 2$, if $[h, i] = 0$, $\{[h + 1, i]\} \# 0 *$, and $[h - 1, i] \neq 0$.

From this definition it may be seen that there is one and only one key

matrix in every vanishing area.* It is the single matrix A in the top row of the vanishing area, provided that this matrix is not in the first column. When the matrix A in the top row of a vanishing area is in the first column, the key matrix is then the matrix A in the row next below and the second column. If $[2, i]$ is a key matrix, we may or may not have $[1, i-1] = 0$.

THEOREM 7. If $[h, i] = 0$ for a set of $n-1$ out of n lines, then $[h, i+1] = 0$ for the set of n lines.

In the matrix

$$\begin{Vmatrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{Vmatrix}$$

for the $n-1$ lines, all the determinants of highest order vanish. Hence in the matrix

$$\begin{Vmatrix} a_h & \cdots & a_{n-i} & t^{i-h+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & \cdots & a_{n-h} & t \\ a_{i+1} & \cdots & a_{n-h+1} & 1 \end{Vmatrix}$$

where the a's are for the n lines, all the determinants of highest order which contain the last column vanish. It follows, by Theorem B, that all the determinants of highest order in the matrix vanish, and hence that $[h, i+1] = 0$ for the n lines.

THEOREM 8. If $[h, i] = 0$ for n lines, and $i \leqq n-p$; then $\{[h, i]\} = 0$ for any set of $n-1$ out of the n lines.

The condition $i \leqq n-p$ is needed to insure the existence of the matrix $[h, i]$ for $n-1$ lines; for $i > n-p$ can only hold when n is odd and $i = p$, in which case the matrix $[h, i]$ would be in the lowest row of the table for n lines, and would not exist for $n-1$ lines.

In the matrix

$$\begin{Vmatrix} a_h & \cdots & a_{n-i+1} \\ \cdot & \cdot & \cdot \\ a_i & \cdots & a_{n-h+1} \end{Vmatrix}$$

all the determinants of highest order vanish. Hence all the determinants of highest order in the matrix

$$\begin{Vmatrix} a_h & \cdots & \cdots & a_{n-i+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & \cdots & \cdots & a_{n-h+1} \\ t^{n-h-i+1} & \cdots & t & 1 \end{Vmatrix}$$

vanish for all values of t. But when t is the clinant of one of the n lines,

* A trivial exception which will not concern us is the vanishing area in which the only matrix A is the matrix $[1, p]$.

the solid determinants of this matrix are (except for a factor which is a power of t) the solid determinants of the matrix

$$\left\|\begin{matrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{matrix}\right\|$$

where the α's are for the remaining $n-1$ lines. Hence $\{[h, i]\} = 0$ for any set of $n-1$ out of the n lines.

THEOREM 9. If $[h, i]$ is a key matrix for n lines, where $i \gtreqless n-p$, then $[h, i]$ is also a key matrix for at least $n+h-i$ of the sets of $n-1$ out of the n lines.

By the preceding theorem, $\{[h, i]\} = 0$ for any set of $n-1$ out of the n lines. Hence for one of these sets of $n-1$ lines (see Theorem 3). either $\{[h+1, i]\} = 0$, or $\{[h+1, i]\} \# 0$, or $h = i$ and $[h+1, i]$ does not exist. Suppose that $\{[h+1, i]\} = 0$ for a certain set of $n-1$ lines, Then

$$\begin{vmatrix} \alpha_{h+1} & \cdots & \alpha_i \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{2i-h+1} \end{vmatrix} = 0$$

for this set, or

$$\begin{vmatrix} a_{h+1} & \cdots & \cdot & a_{i+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & \cdots & \cdot & a_{2i-h} \\ t^{i-h} & \cdots & t & 1 \end{vmatrix} = 0$$

where the a's are for the n lines and t is the clinant of the omitted line. If this last equation held for more than $i-h$ distinct values of t, it would hold for all values of t, and we would have

$$\begin{vmatrix} a_{h+1} & \cdots & a_i \\ \cdot & \cdot & \cdot \\ a_i & \cdots & a_{2i-h-1} \end{vmatrix} = 0.$$

But this is the first solid determinant in the matrix $[h+1, i]$ for n lines, and by hypothesis $\{[h+1, i]\} \# 0$. Hence we can not have $\{[h+1, i]\} = 0$ for more than $i-h$ sets of $n-1$ lines, and therefore we must have $\{[h+1, i]\} \# 0*$ for at least $n+h-i$ sets. But if for a set of $n-1$ lines $\{[h, i]\} = 0$ and $\{[h+1, i]\} \# 0*$, then, by Theorem 4, $[h, i] = 0$. For $h = 2$ this proves our theorem. For $h > 2$, it remains to show that for a set of $n-1$ lines for which $[h, i] = 0$ and $\{[h+1, i]\} \# 0*$, we have also $[h-1, i] \neq 0$. Suppose that $[h-1, i] = 0$. Then, by Theorem 5, $[h-1, i-1] = 0$. Then, by Theorem 7, $[h-1, i] = 0$ for the n lines, which contradicts the hypothesis that $[h, i]$ is a key matrix for the n lines. Hence for our set of $n-1$ lines, $[h-1, i] \neq 0$, and $[h, i]$ is a key matrix.

THEOREM 10. If $[h, i]$ is a key matrix for all sets of $n-1$ out of n lines, then $[h, i] = 0$ for the n lines.

In the matrix

$$\begin{Vmatrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{Vmatrix}$$

for any one of the sets of $n-1$ lines, all the determinants of highest order vanish. Therefore in the matrix

$$\begin{Vmatrix} a_h & & \cdots & \cdot & a_{n-i+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_i & & \cdots & \cdot & a_{n-h+1} \\ t^{n-h-i+1} & & \cdots & t & 1 \end{Vmatrix}$$

where the a's are for the n lines and t is the clinant of any one of them, all the solid determinants vanish. Each one of these solid determinants is of the $(i-h+1)$th degree in t (except for a factor which is a power of t); and since $i-h+1<n$, it must vanish for all values of t. Hence all the solid determinants in

$$\begin{Vmatrix} a_h & \cdots & a_{n-i+1} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ a_i & \cdots & a_{n-h+1} \end{Vmatrix}$$

vanish, or $\{[h, i]\} = 0$ for the n lines. Then either $\{[h+1, i]\} = 0$ or $\{[h+1, i]\} \# 0 *$. Suppose $\{[h+1, i]\} = 0$. Then $[h+1, i]$ would lie in a vanishing area having a certain key matrix; and, by the preceding theorem, some sets of $n-1$ lines would have the same key matrix; and therefore for some sets of $n-1$ lines we would have $\{[h+1, i]\} = 0$. But this is contrary to the hypothesis that $[h, i]$ is a key matrix for all sets of $n-1$ lines. Therefore $\{[h+1, i]\} \# 0 *$ for the n lines, and, by Theorem 4, it follows that $[h, i] = 0$ for the n lines.

§ 5. *Necessary and Sufficient Condition for an Improper Set.*

THEOREM 11. (A) If $[k+2, i]$ is a key matrix for n lines, $n \geqq 3$, $0 \leqq k \leqq p-2$, $k+2 \leqq i \leqq p$; then there exists a unique integer g in the interval $n+k-i+2 \leqq g \leqq p-2$, such that g (and no more) of the n lines touch a unique curve $P_k^{(g-n+k+i-1)}$: and conversely,

(B) If g (and no more) of n lines touch a curve $P_k^{(g-n+k+i-1)}$, $n \geqq 3$, $0 \leqq k \leqq p-2$, $k+2 \leqq i \leqq p$, $n+k-i+2 \leqq g \leqq n$; then $[k+2, i]$ is a key matrix for the n lines.

It has been shown in § 2 that this theorem holds for $n = 3, 4, 5, 6$; and the general theorem may be proved by induction. It will first be shown that (A) holds for n lines if (A) and (B) both hold for $n-1$ lines.

Consider first the case where $i \overline{\gtrless} n-p$, and therefore $[k+2, i]$ exists for any set of $n-1$ out of the n lines. For at least $n+k-i+2$ of these sets of $n-1$ lines, $[k+2, i]$ is a key matrix (Theorem 9). Applying (A) to one of these sets of $n-1$ lines,* it follows that there is an integer g_1, $n+k-i+1 \overline{\gtrless} g_1 \overline{\gtrless} n-1$, such that g_1 of the $n-1$ lines touch a curve $P_k^{(g_1-n+k+i)}$. Similarly, for another set of $n-1$ lines, g_2 of them will touch a curve $P_k^{(g_2-n+k+i)}$. Then at least g_1+g_2-n lines would be tangent to both of these curves, and therefore the curves can not be distinct. For these curves, if distinct, can have at most $g_1+g_2+2i-2n-1$ common tangents (see § 3); and from $p \overline{\gtrless} i$ and $i \overline{\gtrless} n-p$ we find $g_1+g_2-n > g_1+g_2+2i-2n-1$. It follows that there is a unique curve $P_k^{(g_1-n+k+i)}$, $n+k-i+1 \overline{\gtrless} g_1 \overline{\gtrless} n-1$, which is touched by exactly g_1 lines out of each set of $n-1$ lines for which $[k+2, i]$ is a key matrix. If this is true for every set of $n-1$ lines out of the n lines, it follows at once that $g_1 = n-1$, and that all of the n lines touch a curve $P_k^{(k+i-1)}$. If there are sets of lines for which $[k+2, i]$ is not a key matrix, exactly g_1-1 or g_1+1 lines† of such a set must touch the $P_k^{(g_1-n+k+i)}$. In the former case, (B) says that $[k+2, i+1]$ would be a key matrix for the $n-1$ lines, and this would give $\{[k+2, i]\} \neq 0 *$ (Theorem 4). But since $[k+2, i]$ is a key matrix for the n lines, we have $\{[k+2, i]\} = 0$ for all sets of $n-1$ lines (Theorem 8). Hence if there are any sets of $n-1$ lines for which $[k+2, i]$ is not a key matrix, exactly g_1+1 lines of each such set must touch the $P_k^{(g_1-n+k+i)}$. In this case it is evident that exactly g_1+1 of the n lines touch this $P_k^{(g_1-n+k+i)}$, and hence (A) holds for the n lines with $g = g_1+1$.

We must next consider the case $i > n-p$, which only occurs when $n = 2p-1$ and $i = p$. We wish then to show that if $[k+2, p]$ is a key matrix for $2p-1$ lines, $2p-1 \overline{\gtrless} 3$, $0 \overline{\gtrless} k \overline{\gtrless} p-2$; then exactly g of the lines touch a curve $P_k^{(g+k-p)}$, where $p+k+1 \overline{\gtrless} g \overline{\gtrless} 2p-1$.

We have

$$\begin{vmatrix} a_{k+2} & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-2} \end{vmatrix} = 0,$$

and hence

$$\begin{vmatrix} c_1 t_1^{p-1} & \cdots & c_1 t_1^{k+1} & t_1^{p+k-1} & \cdots & t_1 & 1 \\ c_2 t_2^{p-1} & \cdots & c_2 t_2^{k+1} & t_2^{p+k-1} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{2p-1} t_{2p-1}^{p-1} & \cdots & c_{2p-1} t^{k+1} & t_{2p-1}^{p+k-1} & \cdots & t_{2p-1} & 1 \end{vmatrix} = 0.$$

* The inequality conditions of the hypothesis are readily seen to hold for the $n-1$ lines, provided we assume, as of course we may, that $n > 3$.

† Any set of $n-1$ lines must evidently contain either g_1-1, g_1, or g_1+1 lines touching the $P_k^{(g-n+k+i)}$; and it can not be g_1, because this, by (B), would make $[k+2, i]$ the key matrix.

All the minors of the elements of the first column can not vanish. For they are (except for a non-vanishing factor) the determinants

$$\begin{vmatrix} \alpha_{k+3} & \cdots & \alpha_p \\ \cdot & \cdot & \cdot \\ \alpha_p & \cdots & \alpha_{2p-k-3} \end{vmatrix}$$

for the sets of $2p - 2$ out of the $2p - 1$ lines; and if they all vanished we would have, in terms of the constants for the $2p - 1$ lines,

$$\begin{vmatrix} a_{k+3} & \cdots & a_p & a_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-3} & a_{2p-k-2} \\ t^{p-k-2} & \cdots & t & 1 \end{vmatrix} = 0$$

for $2p - 1$ different values of t. This would therefore be an identity in t, and we would have

$$\begin{vmatrix} a_{k+3} & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-3} \end{vmatrix} = 0$$

for the $2p - 1$ lines, contrary to the hypothesis. Suppose the non-vanishing minor to be in the upper right-hand corner. Then the reflex point c and clinant t of each of the $2p - 1$ lines satisfies the equation

$$\text{(I)} \quad \begin{aligned} A_1ct^{p-1} + A_2ct^{p-2} + \cdots + A_{p-k-1}ct^{k+1} + B_1t^{p+k-1} \\ + B_2t^{p+k-2} + \cdots + B_{p+k-1}t + B_{p+k} = 0 \end{aligned}$$

where the A's and B's are the co-factors of the elements of the lower row, and $A_1 \neq 0$. Solving for ct, we have

$$ct = \frac{-B_1t^{p+k-1} - B_2t^{p+k-2} - \cdots - B_{p+k}}{A_1t^{p-2} + A_2t^{p-3} + \cdots + A_{p-k-1}t^k}$$

and multiplying numerator and denominator of the fraction by

$$\frac{(\sqrt{-1})^{p-1}}{(t_1t_2t_3\cdots t_{2p-2})^{(p+k-1)/2}},$$

they both become self-conjugate, and we have

$$\text{(II)} \qquad ct = \frac{\alpha_1t^{p+k-1} + \alpha_2t^{p+k-2} + \cdots + \bar{\alpha}_2t + \bar{\alpha}_1}{\gamma_{k+1}t^{p-2} + \gamma_{k+2}t^{p-3} + \cdots + \bar{\gamma}_{k+1}t^k}$$

where $\gamma_{k+1} \neq 0$.

The reflex point c and clinant t of each of the $2p - 1$ lines must satisfy (II), unless the clinant is such that it causes the denominator and numerator to vanish. If then the numerator and denominator have no common polynomial factor, all of the $2p - 1$ lines satisfy (II), and hence all of

them are tangents of the curve $P_k^{(k+p-1)}$ whose envelope equation is

$$tx + y = \frac{\alpha_1 t^{p+k-1} + \cdots + \bar{\alpha}_1}{\gamma_{k+1} t^{p-2} + \cdots + \bar{\gamma}_{k+1} t^k}$$

which is in accordance with part (A) of our theorem.

Suppose now that the numerator in (II) is not identically zero, and that the numerator and denominator have a highest common factor of degree $\mu + \nu$, containing t^ν (but not $t^{\nu+1}$) as a factor. It is obvious that μ and ν must satisfy the inequalities

$$0 \leqq \mu \leqq p - k - 2, \qquad 0 \leqq \nu \leqq k.$$

We may then take out this common factor and write*

$$\text{(III)} \qquad ct = \frac{\alpha_1' t^{p+k-\mu-2\nu-1} + \cdots + \bar{\alpha}_1'}{\gamma_{k-\nu-1}' t^{p-\mu-\nu-2} + \cdots + \bar{\gamma}_{k-\nu-1}' t^{k-\nu}}.$$

Each of the $2p - 1$ lines, satisfying (I), must satisfy (III) unless its clinant is such as to cause the removed common factor to vanish. Since the clinant of a line can not be zero, it follows that at least $2p - \mu - 1$ of them satisfy (III). Say that exactly $2p - \mu + \sigma - 1$ of the lines satisfy (III), $0 \leqq \sigma \leqq \mu$. These $2p - \mu + \sigma - 1$ lines evidently touch a curve $P_{k-\nu}^{(p+k-\mu-2\nu-1)}$. Then either all of the $2p - 1$ lines touch this curve, and therefore by (B) $[k - \nu + 2,\ p - \sigma - \nu]$ is a key matrix for every set of $2p - 2$ lines; or in at least one set of $2p - 2$ lines exactly $2p - \nu + \sigma - 1$ lines touch the curve, and hence $[k - \nu + 2,\ p - \sigma - \nu - 1]$ is a key for this set of $2p - 2$. In either case it follows that $[k - \nu + 2,\ p - \sigma - \nu] = 0$ for the $2p - 1$ lines (by Theorem 10 or 7). With $[k - \nu + 2,\ p - \sigma - \nu] = 0$, $\sigma \geqq 1$ would give (Theorem 1) $[k + 2,\ p - 1] = 0$, contrary to hypothesis; and hence we must have $\sigma = 0$. With $\sigma = 0$, $\nu \geqq 1$ would give $[k + 1, p] = 0$; and since $\nu \geqq 1$ would necessitate $k \geqq 1$, this again contradicts the hypothesis that $[k + 2, p]$ is a key matrix. It follows that in equation (III) we must have $\nu = 0$, and that the corresponding envelope equation

$$tx + y = \frac{\alpha_1' t^{p+k-\mu-1} + \cdots + \bar{\alpha}_1'}{\gamma_{k-1}' t^{p-\mu-2} + \cdots + \bar{\gamma}_{k-1}' t^k}$$

represents a curve $P_k^{(p+k-\mu-1)}$ which is touched by exactly $2p - \mu - 1$ of the $2p - 1$ lines. This is in agreement with part (A) of our theorem.

There remains the case where the numerator in equation (II) vanishes identically. In this case, at least $2p - 1 - (p - k - 2)$, or $p + k + 1$, of the $2p - 1$ lines are concurrent at the origin, *i.e.*, touch a $P_0^{(1)}$. Say

* The assumption that t^ν is a factor of the numerator gives $\alpha_1 = \alpha_2 = \cdots = \alpha_\nu = 0$; and when we then divide out t^ν, the degree of the numerator is reduced by 2ν.

that exactly $p + k + \sigma + 1$ of them are concurrent. As in the preceding paragraph, a contradiction will result unless $k = \sigma = 0$. If then the numerator in (II) vanishes identically, exactly $p + 1$ of the $2p - 1$ lines must touch a curve $P_0^{(1)}$.

This completes the proof that if (A) and (B) hold for $n - 1$ lines, then (A) must hold for n lines. It will now be shown that if (A) and (B) hold for $n - 1$ lines and (A) holds for n lines, then (B) also holds for n lines.

If g out of n lines touch a curve $P_k^{(g-n+k+i-1)}$, then in g of the sets of $n - 1$ out of the n lines, $g - 1$ lines touch the curve, while in the remaining $n - g$ sets, g lines touch the curve. Hence, applying (B) to the sets of $n - 1$ lines, $[k + 2, i]$ is a key matrix for g of the sets and $[k + 2, i - 1]$ for the remaining $n - g$ sets. It follows (by Theorem 7 if $g < n$, and by Theorem 10 if $g = n$) that $[k + 2, i] = 0$ for the n lines. There must then be a key matrix $[h', i']$ for the n lines, such that $i' + h' \gtreqless k + i + 2$ and $i' - h' \gtreqless i - k - 2$. Hence, applying (A) to the n lines, and proceeding as above, we have that either $[h', i']$ is a key matrix for all of the sets of $n - 1$ lines or $[h', i' - 1]$ is a key matrix for some sets of $n - 1$ lines. In the former case there would be a set of $n - 1$ lines with both $[h', i']$ and $[k + 2, i]$ as key matrices; while in the latter case there would be a set of $n - 1$ lines with key matrices $[h', i' - 1]$ and either $[k + 2, i]$ or $[k + 2, i - 1]$. Either case is a contradiction unless $[h', i']$ is $[k + 2, i]$. It follows that if g of the lines touch a curve $P_k^{(g-n+k+i-1)}$, $[k + 2, i]$ is a key matrix. If then (A) and (B) hold for $n - 1$ lines, they both hold for n lines; and this establishes our theorem.

THEOREM 12. If $[k + 2, i]$ is a key matrix for n lines, then for g of the sets of $n - 1$ out of the n lines (where $n + k - i + 2 \gtreqless g \gtreqless n$) $[k + 2, i]$ is a key matrix, and for the remaining $n - g$ sets $[k + 2, i - 1]$ is a key matrix.

This theorem is an obvious corollary of Theorem 11.

As noted at the beginning of § 3, a set of $2p$ lines is improper when, and only when, the determinant

$$\begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}$$

vanishes. Since this determinant is one of the two complementary solid determinants in the matrix $[2, p]$, an improper set will occur when, and only when, $\{[2, p]\} = 0$. For $2p - 1$ lines to be improper, either or both of the conditions $[2, p] = 0$, $[3, p] = 0$, must hold. And since these vanishing matrices must lie in a vanishing area having a key matrix, it is evident that the necessary and sufficient condition for an improper set is

that there should be a key matrix [h, i] somewhere in the area $h \geqq 2$, $i \leqq p$, $i + h \leqq p + 3$, and $i - h \geqq 0$. Since vanishing areas can not overlap, it is evident that there could not be more than one key matrix in this area. And from Theorem 11 we then have

THEOREM 13. A necessary and sufficient condition for an improper set of n lines is that there should exist a unique set of integers k, i, and g, satisfying the inequalities

$$0 \leqq k \leqq p - 2, \qquad k + i + 2 \leqq p + 3,$$

$$k + 2 \leqq i \leqq p, \qquad n + k - i + 2 \leqq g \leqq n,$$

and such that g (and no more) of the n lines are tangent to a unique curve $P_k^{(g-n+k+i-1)}$.

§ 6. *Improper Sets as Limiting Cases.*

A set of n variable lines with characteristic constants $\alpha_1, \alpha_2, \cdots, \alpha_n$ and clinants $\tau_1, \tau_2, \cdots, \tau_n$ will be said to approach a set of n fixed lines with constants $a_1, a_2, \cdots, a_n$ and clinants $t_1, t_2, \cdots, t_n$, if the α's and τ's simultaneously approach the corresponding a's and t's. Since there are only a finite number of conditions for improper sets, it is always possible (in an infinite number of ways) to find a sequence of proper sets which approach any given improper set. The question then arises as to whether, when a variable proper set thus approaches a given improper set, the Clifford figure (point or circle) of the variable set will approach, under any circumstances, a unique limit which might be conveniently defined as the Clifford figure of the improper set. This question will be answered without giving the rather tedious details of the proofs. The results may be most conveniently expressed if we assume for the time being the point of view of the geometry of inversion; *i.e.*, we assume that the plane has only one point at infinity, and understand the word circle to include straight lines and point-circles (circles of zero radius).

Type 1; $2p - 1$ lines, key matrix [$k + 2$, $p - k + 1$], $k \geqq 1$. This key matrix lies on the skew diagonal $h + i = p + 3$; and g of the $2p - 1$ lines, $p + 2k \leqq g \leqq 2p - 1$, touch a curve $P_k^{(g-p+1)}$. In the fractions giving the center and radius of the Clifford circle, the denominators vanish while the numerators do not. Of the sets of $2p - 2$ out of the $2p - 1$ lines, g sets are proper sets; but the remaining $2p - g - 1$ sets are of the type 4 described below, with their Clifford points at infinity. In this case the g finite Clifford points lie on a straight line.* When a proper set of $2p - 1$ lines approaches such an improper set in any way, the Clifford circle approaches this straight line as a limit. We may define this line therefore as the Clifford figure in this case. The reflex point of this line is

* See Morley, loc. cit.

$$c = \begin{vmatrix} a_1 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} \end{vmatrix} \bigg/ \begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}.$$

The denominator of this fraction is a solid determinate of the matrix $[1, p-1]$, and can not vanish, since with $[k+2, p-k+1]$ as a key matrix we have $\{[1, p-1]\} \neq 0$. As examples of sets of this type we have 5 lines touching a deltoid; 7 lines touching a $P_2^{(4)}$; 7 lines, 6 of which touch a deltoid; etc.

Type 2; $2p-1$ lines, key matrix $[k+2, p-k]$, $k \geqq 1$. The key matrix lies on the skew diagonal $h+i=p+2$; and g of the $2p-1$ lines, $p+2k+1 \leqq g \leqq 2p-1$, touch a curve $P_k^{(g-p)}$. Both numerators and denominators of the fractions in the expression for the Clifford circle vanish. Of the sets of $2p-2$ lines, g sets are of the type 4 and the rest of the type 5 described below, with their Clifford points at infinity in both cases. When a proper set of lines approaches a set of this kind in any way, the entire Clifford circle recedes to infinity. We may therefore define the Clifford figure for this type to be a point-circle at infinity. The simplest example of a set of this type is the case of 7 lines touching a deltoid.

Type 2′; $2p-1$ lines, key matrix $[2, p]$. In this type, g of the lines, $p+1 \leqq g \leqq 2p-1$, touch a curve $P_0^{(g-p)}$; and the radius of the Clifford circle is zero.* Of the sets of $2p-2$ lines, g sets are proper, and the rest are of type 5′ with a finite Clifford point as defined below. All of these $2p-1$ Clifford points for the sets of $2p-2$ lines coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-3} \end{vmatrix} \bigg/ \begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix},$$

which is the focus† of the curve $P_0^{(g-p)}$. When a variable proper set approaches such a set, the center of the Clifford circle approaches this point while its radius approaches zero. We may define the Clifford figure for this type to be a point-circle at this point.

Type 3; $2p-1$ lines, key matrix $[k+2, p-k-1]$, $k \geqq 1$. This key matrix lies on the skew diagonal $h+i=p+1$; and g of the lines, $p+2k+2 \leqq g \leqq 2p-1$, touch the curve $P_k^{(g-p-1)}$. Here again both the numerators and denominators of the fractions for the center and radius of the Clifford circle vanish. Of the sets of $2p-2$ lines, g sets are of type 5 with their Clifford points at infinity, and the rest are of type 6 with entirely indeterminate Clifford points. Given a set of this type, a proper set may be made to approach it in such a way that the Clifford circle will approach any arbitrarily preassigned straight line. The simplest example of this type is a set of 9 lines touching a deltoid.

* See note, beginning of § 2.

† See Clifford, loc. cit. The denominator of this fraction can not vanish.

Type 3′; $2p - 1$ lines, key matrix $[2, p - 1]$. In this case g of the lines, $p + 2 \leqq g \leqq 2p - 1$, touch a curve $P_0^{(g-p-1)}$; and again the Clifford circle fractions take the indeterminate form. Of the sets of $2p - 2$ lines, g sets have a Clifford point as defined in type 5′, while the remaining sets are of type 6 with indeterminate Clifford points. The g Clifford points coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-2} \\ \cdot & \cdot & \cdot \\ a_{p-2} & \cdots & a_{2p-5} \end{vmatrix} \Big/ \begin{vmatrix} a_3 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-5} \end{vmatrix},$$

which is the focus of the curve $P_0^{(g-p-1)}$. As a proper set of lines approaches a set of this type, the circumference of the Clifford circle will always approach this point. We may then say that the Clifford circle for this type is indeterminate except that it must pass through this point. For example, if a set of 7 lines touch a parabola, a proper set may be made to approach these 7 lines in such a way that the Clifford circle will approach any preassigned circle or straight line through the focus of the parabola, or a point-circle at the focus. A still simpler example is that of 5 concurrent lines.

Type 4; $2p$ lines, key matrix $[k + 2, p - k + 1]$, $k \geqq 1$. The key matrix is on the skew diagonal $h + i = p + 3$; and g of the lines, $p + 2k + 1 \leqq g \leqq 2p$, touch a curve $P_k^{(g-p)}$. In the fraction representing the Clifford point the denominator vanishes while the numerator does not. Of the sets of $2p - 1$ out of the $2p$ lines, g sets are of type 1 and have Clifford lines, while the remaining sets are of type 2 having Clifford point-circles at infinity. As a proper set approaches a set of this type, the Clifford point moves off to infinity; and hence one may define the point at infinity to be the Clifford point for this type. The simplest example is a set of 6 lines touching a deltoid.

The g Clifford lines for the g sets of $2p - 1$ lines in this case exhibit a very interesting property which is treated in the next section.

Type 5; $2p$ lines, key matrix $[k + 2, p - k]$, $k \geqq 1$. This key matrix lies in the skew diagonal $h + i = p + 2$; and g of the lines, $p + 2k + 2 \leqq g \leqq 2p$, touch a curve $P_k^{(g-p-1)}$. The numerator and denominator of the fraction giving the Clifford point both vanish. Of the sets of $2p - 1$ lines, g are of type 2 and the rest of type 3, having respectively the point at infinity and indeterminate straight lines for their Clifford circel. An iso type 4, we may define the Clifford point for this type to be the pinst at infinity. The simplest example is a set of 8 lines tangent to a deltoid.

Type 5′; $2p$ lines, key matrix $[2, p]$. Here g of the lines, $p + 2 \leqq g \leqq 2p$, touch a curve $P_0^{(g-p-1)}$, and the expression for the Clifford point

takes the indeterminate form. Of the sets of $2p-1$ lines, g sets are of type $2'$ and the rest of type $3'$; the former having Clifford point-circles and the latter indeterminate circles passing through fixed points. These points for all of the sets of $2p-1$ lines coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-8} \end{vmatrix} \Bigg/ \begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix},$$

which is the focus of the curve $P_0^{(g-p-1)}$. When a proper set approaches a set of this type in any way, the Clifford point will always approach this point of coincidence; and it may therefore be defined as the Clifford point for a set of this type. As simple examples we have 4 concurrent lines; 6 lines, 5 of which are concurrent; 6 lines touching a parabola; etc.

Type 6 consists of all improper sets not included in the preceding types. The key matrix will lie somewhere above the skew diagonal $h+i=n-p+2$. In the fractions in the expressions for the Clifford figure, both numerator and denominator will vanish. Given such an improper set of lines, one may arbitrarily preassign a Clifford figure, and then cause a proper set to approach the improper set in such a way that its Clifford figure will approach the preassigned figure. The Clifford figure for this type is therefore entirely indeterminate.

To summarize, we may say that a unique Clifford figure exists for every proper set, and, as defined in this section, for every improper set of types 1, 2, $2'$, 4, 5, and $5'$; that it is partially indeterminate for sets of type 3 and $3'$; and that it is entirely indeterminate for sets of type 6. The existence in this sense of a uniquely defined Clifford figure for a set of n lines does not imply (even for a proper set) the existence of such a determinate Clifford figure for every sub-set of lines. But it does imply

(1) That a determinate Clifford figure exists for at least $p+1$ of the sets of $n-1$ lines;

(2) That these Clifford figures which exist for sets of $n-1$ lines are all incident with the Clifford figure for the n lines;

(3) That when a variable proper set of n lines approaches the given set in any way, the Clifford figure of the variable set will always approach that of the given set.

§ 7. *The Reciprocal Case.*

In type 4 of the preceding section, g of the $2p$ lines touch a curve $P_k^{(g-p)}$, and g of the sets of $2p-1$ lines (the sets obtained by omitting one by one the g lines which touch the curve) have Clifford lines instead of circles. The reflex point of one of these lines (see type 1) is

$$c = \begin{vmatrix} a_1 & \cdots & a_p & t^p \\ \cdot & \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} & t \\ a_{p+1} & \cdots & a_{2p} & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} a_2 & \cdots & a_{p+1} & t^{p-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} & t \\ a_{p+2} & \cdots & a_{2p} & 1 \end{vmatrix}$$

where the a's are for the $2p$ lines and t is the clinant of the omitted line. With $[k+2,\ p-k+1]$ as key matrix, the coefficients of the highest and lowest powers of t in the denominator vanish, so that we may write

$$c = \frac{A_p t^p + A_{p-1}t^{p-1} + \cdots + A_1 t + A_0}{B_{p-k}t^{p-k} + B_{p-k-1}t^{p-k-1} + \cdots + B_{k+1}t^{k+1}}$$

where, moreover, $A_p \neq 0$ and $B_{p-k} \neq 0$. The clinant τ of this line is equal to the quotient $\bar{c}/c$. Under the condition $[k+2,\ p-k+1] = 0$, it may be seen that the ratio $B_s/\bar{B}_{p-s+1}$ is a constant turn for all values of s; and if we therefore put $B_s/\bar{B}_{p-s+1} = T$, we find

$$\tau = \frac{\bar{c}}{c} = (-S)^p T t,$$

where S is the product of the $2p$ clinants of the lines of the set. We may then put $t = \tau/(-S)^p T$, and thus obtain an expression for the reflex point c of the line in terms of its clinant τ, thus

$$c = \frac{C_p t^p + C_{p-1}t^{p-1} + \cdots + C_1 t + C_0}{D_{p-k}t^{p-k} + \cdots + D_{k+1}t^{k+1}},$$

where

$$C_j = A_j/(-S)^{pj}T^j \qquad \text{and} \qquad D_s = B_s/(-S)^{ps}T^s.$$

If we now multiply both sides of the equation by τ, and the numerator and denominator of the fraction by $(-S)^{(p^2+p)/2}T^{p/2}$, we obtain

$$c\tau = \frac{\alpha_1\tau^p + \alpha_2\tau^{p-1} + \cdots + \bar{\alpha}_2\tau - \bar{\alpha}_1}{\gamma_{k+1}\tau^{p-k-1} + \cdots + \bar{\gamma}_{k+1}\tau^k},$$

where the numerator and denominator are now found to be self conjugate, and $\alpha_1 \neq 0$ and $\gamma_{k+1} \neq 0$. Hence we see that these g Clifford lines are tangents of the curve $P_k^{(p)}$ whose envelope equation is

$$tx + y = \frac{\alpha_1\tau^p + \alpha_2\tau^{p-1} + \cdots + \bar{\alpha}_1}{\gamma_{k+1}\tau^{p-k-1} + \cdots + \bar{\gamma}_{k+1}\tau^k}.$$

The most interesting case occurs when $g = 2p$, *i.e.*, when all of the $2p$ lines touch a curve $P_k^{(p)}$. We then obtain $2p$ Clifford lines for the sets of $2p-1$ lines, and these Clifford lines touch a new curve $P_k^{(p)}$. An examination of the incidence relations of the various Clifford points and circles for the subsets of lines shows that the relation between the two

sets of $2p$ lines is reciprocal; *i.e.*, if we start with the $2p$ Clifford lines, we are led back to the original set.*

There is also a certain reciprocity in the cases where $g < 2p$. For instance, when $g = 2p - 1$, $2p - 1$ of the $2p$ lines touch a $P_k^{(p-1)}$, and we are led to $2p - 1$ Clifford lines touching a $P_k^{(p)}$. Reversing the process, and starting with these $2p - 1$ lines, we have a case of type 1 with a Clifford line, this line being the one line of the original set which does not touch the $P_k^{(p-1)}$. For $g < 2p - 1$, the fact that the g Clifford lines touch a $P_k^{(p)}$ is not sufficient to make them an improper set;† and hence they will, in general, be a proper set and have a Clifford element which will be the Clifford element of the $2p - g$ lines of the original set which do not touch the $P_k^{(g-p)}$.

For example, suppose that 19 of 24 lines touch a curve $P_3^{(7)}$, while the remaining 5 lines have no exceptional conditions upon them. Then 19 of the sets of 23 lines will be of type 1 and have Clifford lines, and these 19 Clifford lines will touch a $P_3^{(12)}$. Starting with these 19 lines, they will be found to be a proper set, their Clifford circle being that of the 5 lines of the original set which did not touch the $P_3^{(7)}$. But if the 5 lines of the original set touch a parabola, and hence have a point-circle as their Clifford figure, then the 19 lines in the reverse process will lead to this same point-circle as their Clifford figure, and will therefore be of type 2′; and it follows that in addition to the fact that the 19 Clifford lines touch a curve $P_3^{(12)}$, g of them, where $11 \leqq g \leqq 19$, must also touch a curve $P_0^{(g-10)}$.

§ 8. *The Curve $P_k^{(m)}$ as a Sum of Fundamental Curves.*

Let two curves $P_{k_1}^{(m_1)}$ and $P_{k_2}^{(m_2)}$ be given by the envelope equations

$$tx + y = \frac{\alpha_1(t)}{\gamma_1(t)} \qquad \text{and} \qquad tx + y = \frac{\alpha_2(t)}{\gamma_2(t)};$$

and suppose that $\gamma_1(t)$ and $\gamma_2(t)$ have no common polynomial factor, which necessitates that one of the k's, say k_2, should be zero. Then the envelope equation

$$tx + y = \frac{\alpha_1(t)}{\gamma_1(t)} + \frac{\alpha_2(t)}{\gamma_2(t)} = \frac{\alpha_1(t)\gamma_2(t) + \alpha_2(t)\gamma_1(t)}{\gamma_1(t)\gamma_2(t)}$$

represents a curve $P_{k_1}^{(m_1+m_2-1)}$ which may be called the "sum" of the two curves. Since the map equations of the three curves $P_{k_1}^{(m_1)}$, $P_0^{(m_2)}$, and $P_{k_1}^{(m_1+m_2-1)}$ are respectively

* Kantor (loc. cit.) discovered this property for a set of 6 lines touching a deltoid. Morley generalized Kantor's theorem; but his generalization is quite different from the one here given.

† For $g \leqq 2p - 2k$, the conclusion that the g lines touch a curve $P_k^{(p)}$ is obviously trivial.

$$x = \frac{d}{dt}\left(\frac{\alpha_1(t)}{\gamma_1(t)}\right), \quad x = \frac{d}{dt}\left(\frac{\alpha_2(t)}{\gamma_2(t)}\right), \quad \text{and} \quad x = \frac{d}{dt}\left(\frac{\alpha_1(t)}{\gamma_1(t)}\right) + \frac{d}{dt}\left(\frac{\alpha_2(t)}{\gamma_2(t)}\right);$$

and since in all three cases the parameter t is the clinant of the tangent to the curve at the point x, the geometric significance of the addition is obvious. If we take two points, one on the curve $P_{k_1}^{(m_1)}$ and the other on the curve $P_0^{(m_2)}$, where the tangents to the curves are parallel, and add these points by the ordinary construction for the sum of two complex numbers; the result will be the point of the curve $P_{k_1}^{(m_1+m_2-1)}$ at which the tangent has the same direction.* We may similarly define the sum of any number of such curves, all the denominators in the envelope equations being prime to each other.

We may call a curve $P_k^{(m)}$ a "fundamental curve" when the denominator $\gamma(t)$ in the envelope equation is a power of a self-conjugate factor of the first degree or a power of a self-conjugate factor of the second degree which is not the product of self-conjugate factors of the first degree. For such a fundamental curve we have $k = 0$, with $\gamma(t)$ of the form $(\gamma t + \overline{\gamma})^{m-1}$ or $(\gamma t^2 + rt + \overline{\gamma})^{(m-1)/2}$; or† $k = \frac{1}{2}(m-1)$, with $\gamma(t)$ of the form rt^k. The map equations in the three cases may be written respectively in the forms

$$x = A_0 + \frac{B_2}{(\gamma t + \overline{\gamma})^2} + \frac{B_3}{(\gamma t + \overline{\gamma})^3} + \cdots + \frac{B_m}{(\gamma t + \overline{\gamma})^m},$$

$$x = A_0 + \frac{B_{m-1}t^{m-1} + B_{m-2}t^{m-2} + \cdots + B_1 t + B_0}{(\gamma t^2 + rt + \overline{\gamma})^{m+1/2}},$$

$$x = A_k t^k + A_{k-1}t^{k-1} + \cdots + A_1 t + A_0 + \frac{B_1}{t} + \frac{B_2}{t^2} + \cdots + \frac{B_{k+1}}{t^{k+1}}.$$

In the first two cases the point A_0 is the focus of the curve; while in the third case it may be called the center (suggested by the center of the deltoid when $k = 1$). When $A_0 = 0$, *i.e.*, when the focus or center is at the origin, the curve may be said to be in canonical position. Every curve $P_k^{(m)}$ is the sum of q such fundamental curves,‡ where q is the number of distinct irreducible self-conjugate factors of $\gamma(t)$. It is evident that if we had such a decomposition of a curve $P_k^{(m)}$ into a sum of fundamental curves, the A_0's for the several fundamental curves would be arbitrary except that their sum would be fixed. The point represented by the sum of the A_0's may be called (for lack of a better name) the center of the $P_k^{(m)}$; and the $P_k^{(m)}$ may be said to be in canonical position when its center is at the origin. If a curve $P_k^{(m)}$ is in canonical position, its decomposition into the sum of fundamental curves in canonical position is unique.

* These curves have one and only one tangent in a given direction.

† This case $k = \frac{1}{2}(m-1)$ is the curve $\Delta^{(2n-1)}$ of Morley and Stephens (loc. cit.).

‡ Theorems 5 and 6 of Atchison's paper (loc. cit.) are special cases of this.

ON THE REPRESENTATIONS OF NUMBERS AS SUMS OF 3, 5, 7, 9, 11 AND 13 SQUARES.*

By E. T. Bell.

I. Preliminary Considerations.

1. Of the nine sections into which this paper is divided, I is preliminary; II contains eleven general formulas of a simple nature; III–VIII apply the first seven formulas of II to the determination in finite form of the number of representations of an integer as a sum of 3, 5, 7, 9, 11 and 13 squares respectively, attention being paid to the numbers of odd squares in the representations; and IX is devoted to recurrences, consequences of the last four formulas of II, for facilitating the numerical computations implied in III–VIII. A complete system of results of a well-defined kind, described in §§ 4–6, is obtained in III–VIII; but in the sense of determining finitely the total number of representations for any linear form of the integer to be represented, the enumerations for 11 and 13 squares are only partial. If complete finite systems are desired in these and further odd cases, they may be found by applying II to the theorems of H. J. S. Smith, Glaisher, and others, concerning an even number of squares. The resulting formulas, however, are simple neither in the common meaning nor in the technical sense defined in § 4; and as they belong to a wholly different order of ideas, they are not included here.

From one point of view the formulas, particularly the recurrences, for 5, 7, 9, 11 and 13 squares may be looked upon as analogues of the class number relations of Kronecker,† which, as has long been known, are intimately related to the decompositions of integers into sums of three squares. In fact, if the incomplete primitives of § 8, which have been expressly avoided here, are admitted, all of Kronecker's results and more of the same kind due to later writers reappear; and it appears that the 5, 7, 9, 11, 13 square theorems are of the same general nature. To find closer analogues it is only necessary to replace the functions denoting the numbers of representations as sums of 5, 7, $\cdots$ squares by their weight equivalents as determined by H. J. S. Smith in his memoir "On the Orders and Genera of Quadratic Forms Containing More than Three Indeterminates." The re-

* Read before the San Francisco Section of the American Mathematical Society, April 5, 1919.

† The development of these and other relations from the present point of view will be published elsewhere.

sults are most numerous for 9 squares. The recurrences evaporate with the case of 11 squares. Taken together, all form a short natural chapter in arithmetic.

In counting representations we include, as customary, both the order of the squares and the signs of their square roots. Thus the single decomposition of 54 into a sum of 9 squares,

$$54 = 1^2 + 2^2 + 2^2 + 3^2 + 3^2 + 3^2 + 3^2 + 3^2 + 0^2,$$

contributes $2^8 \cdot 9!/2!5! = 387072$ representations.

As constant use is made of Glaisher's work for 2, 4, $\cdots$, 12 squares, we may cover all references to it by this citation: *Quarterly Journal*, 38 (1906–7), pp. 1–68. A convenient précis of the square-theorem results of his paper is given in the *Proceedings of the London Mathematical Society*, (2) 5 (1907), pp. 479–490.

2. All letters m, μ, n, a, b, c, r, s, denote positive integers, different from zero unless explicitly noted to the contrary; m, μ are always odd, and the rest, unless further specified, arbitrary; k is an integer $\geqq 0$. We define $f(n)$ to be primitive if its values may be calculated in finite form from the real divisors of n alone, and consider $f(x) = 0$ (or, if preferred, non-existent), when $x \leqq 0$, or when x is not an integer. With these conventions, the respective sums

$$f(n-4) + f(n-16) + f(n-36) + f(n-64) + \cdots,$$

$$f(n-1) + f(n-9) + f(n-25) + f(n-49) + \cdots,$$

consist of only a finite number of terms, and may be written $\Sigma f(n - 4a^2)$, $\Sigma f(n - \mu^2)$, the Σ extending only to those values of a, μ that render $n - 4a^2$, $n - \mu^2 > 0$. Similarly for sums of functions of the triangular numbers, 1, 3, 6, 10, $\cdots$, such as

$$f(n-4) + f(n-12) + f(n-24) + f(n-40) + \cdots,$$

which may be denoted by $\Sigma f(n - 4t)$, t representing, as always throughout the paper, a triangular number.

By repeated application of the obvious identity

$$\Sigma f(n - a^2) = \Sigma[f(n - \mu^2) + f(n - 4a^2)],$$

we get a transformation which frequently is useful:

$$\Sigma f(n - a^2) = \Sigma\Big[f(n - 4^s a^2) + \sum_{r=0}^{s-1} f(n - 4^r \mu^2)\Big].$$

3. We shall require the primitives: $\zeta_g(n)$, = the sum of the gth powers of all the divisors of n; $\zeta_g'(n)$, $\zeta_g''(n)$ the like for the odd, even divisors

respectively; $\xi_g(n)$, = the excess of the sum of the gth powers of all those divisors of n that are $\equiv 1 \bmod 4$ over the like sum for the divisors $\equiv -1 \bmod 4$; $\xi'_g(n)$, = the excess of sum of the gth powers of all those divisors of n whose conjugates are $\equiv 1 \bmod 4$ over the like sum for the divisors whose conjugates are $\equiv -1 \bmod 4$; $\epsilon(n) = \epsilon(a^2n)$, = 1 or 0 according as n is or is not a square. From these we construct further primitives as required. When $g = 0$, $\zeta_g, \cdots, \xi'_g$ are written $\zeta, \cdots, \xi'$ respectively, and denote the numbers of divisors in the respective classes defined by the corresponding functions. Our notation follows Liouville's, to conform with other papers on similar topics. Glaisher's notation is given loc. cit., pp. 3–6.

4. If $f(n)$ is primitive, then a sum of the form $\Sigma f[(pn - qa^2)/g]$, in which p, q, g are numerical constants, n is an integer, and the summation is with respect to the variable integer a, is defined to be simple. Thus, $\Sigma f(n - a^2)$ is a simple sum, and its value being determined when n is given, we shall call $\Sigma f(n - a^2)$ a simple function of n; and, by a legitimate extension, say that any linear function of a finite number of simple functions of n is a simple function of n. The simple functions most frequently occurring hereafter are of the forms $\Sigma f[(m - \mu^2)/g]$, where $g = 1, 2, 4$ or 8; $\Sigma f(n - a^2)$; $\Sigma f(m - 4a^2)$.

5. Let $N_r(n)$, $N_r(n, g)$ denote respectively the total number of representations of n as a sum of r squares, and the total number of representations of n as a sum of r squares precisely g of which are odd. Then clearly $N_r(n)$ is expressible in the form

$$\sum_{g=0}^{r} c_g N_r(n, g),$$

wherein $c_g = 1$ or 0. When the linear form, modulo s, of n is assigned, the constants c_g are known. We shall use $s = 8$; hence, observing that $m^2 \equiv 1 \bmod 8$, $(2a)^2 \equiv 0$ or $4 \bmod 8$, we have for the determination of g,

$$g + \alpha(r - g) \equiv n \bmod 8, \qquad (\alpha = 0, 4); \qquad 0 \leqq g \leqq r.$$

When r is specified we take in this way a census of the possible linear forms modulo 8. For a particular r the census is most readily found by inspection. Thus, for $r = 11$, (cf. the m, n, k-notation in § 2):

$$\begin{aligned}
n &= 4k &&: N_{11}(n) = N_{11}(n, 0) + N_{11}(n, 4) + N_{11}(n, 8),\\
m &= 4k + 1 &&: N_{11}(m) = N_{11}(m, 1) + N_{11}(m, 5) + N_{11}(m, 9),\\
n &= 2m &&: N_{11}(n) = N_{11}(n, 2) + N_{11}(n, 6) + N_{11}(n, 10),\\
m &= 8k + 3 &&: N_{11}(m) = N_{11}(m, 3) + N_{11}(m, 7) + N_{11}(m, 11),\\
m &= 8k + 7 &&: N_{11}(m) = N_{11}(m, 3) + N_{11}(m, 7).
\end{aligned}$$

Hence, for example, knowing $N_{11}(m, 1)$, $N_{11}(m, 5)$, $N_{11}(m, 9)$ when $m \equiv 1$ or 5 mod 8, we can write down the value of $N_{11}(m)$. The census also shows what values of $N_r(n, g)$ vanish; thus, $N_{11}(2m, 8) = 0$, $N_{11}(2m, 4) = 0$.

6. We shall seek to determine all n, g such that

$$N_r(n), \qquad N_r(n, g), \qquad (r = 3, 5, 7, 9, 11, 13)$$

are simple functions of n, and to exhibit a set of simple functions giving the values of $N_r(n)$, $N_r(n, g)$ in these cases. It will appear that there is not a unique set, for the functions in any set may be transformed in many ways by means of the elementary identities existing between the primitives of § 3. On equating different expressions of the same $N_r(n)$ or $N_r(n, g)$ we get, in several instances, rapid recurrences for the successive calculation of the primitives involved; and in all cases the formulas obtained are well adapted to numerical computation. We may state here, reserving full discussion for another occasion, that when linear forms only are considered, $N_r(n)$is simple for no n when $r = 15, 17, 19, 21, 23, 25$, and probably for no odd $r > 25$; the same applies to $N_r(n, g)$; so that the formulas of this paper form a natural class.

7. To compare the expressions through simple functions with the results given by the classical theory, let us consider, from the standpoint of their adaptability to numerical computation, the three following, of which (A) is due to Eisenstein,* (B) to Stieltjes,† and (C) is found in section IV.

$$(A) \qquad \lambda = 8k + 5: \qquad N_5(\lambda) \quad = -112 \sum_{s=1}^{[\lambda/2]} (s|\lambda)s;$$

$$(B) \qquad m = 8k + 5: \qquad N_5(m, 5) = 32\Sigma\zeta_1\left(\frac{m - \mu^2}{4}\right);$$

$$(C) \qquad m = 8k + 5: \qquad N_5(m) \quad = 112\Sigma\zeta_1\left(\frac{m - \mu^2}{4}\right).$$

In (A), λ is divisible by no square > 1; $[x]$ is the greatest integer in x, and $(s|\lambda)$ is the Legendre-Jacobi symbol, $(s|\lambda) = 0$ when s, λ are not relatively prime. In (B), (C), m is unrestricted, and the summation refers, by the conventions of § 2, to all odd μ such that $0 < \mu \leqq [\sqrt{m}]$. Let $m = 133$; the computation by (A) requires as a first step the calculation of the quadratic characters with respect to 133 of the 55 numbers 1, 2, 3, $\cdots$, 66 prime to 133 and < 67. For a large λ the s prime to λ would have to be determined in practically the same way as the $(s|\lambda)$, viz., by Eisenstein's or one of the equivalent algorithms for $(s|\lambda)$, which amounts to the

* Eisenstein, Crelle, 35 (1847), p. 368.

† Stieltjes, *Comptes Rendus*, 97 (1883), p. 981.

conversion of the s/λ into continued fractions, so that for λ not factorable by inspection there would be in all $[\lambda/2]$ such calculations. In that all of these may be performed non-tentatively, formulas such as (A) are superior to (B) or (C). But with the aid of a factor table, it would seem that even for λ fairly large, say $\lambda = 10005$, the 50 resolutions into prime factors required by (C) could be performed more expeditiously than the (approximately) 2500 conversions necessary in (A). Again, if it were required to construct tables, we should have the advantage of recurrences (found in section IX), such as

$$m = 8k + 5: \qquad N_5(m) = -14\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m - 8t),$$

and several similar relations between the primitives ξ_2 whereby their computation may be greatly abridged. For $m = 133$, we find from (C),

$$\begin{aligned} N_5(133) &= 112[\zeta_1(33) + \zeta_1(31) + \zeta_1(27) + \zeta_1(21) + \zeta_1(13) + \zeta_1(3)] \\ &= 112[48 + 32 + 40 + 32 + 14 + 4] = 19040. \end{aligned}$$

8. Functions that may be calculated from the divisors of n subject to one or more conditions were called by Hermite incomplete. Formulas analogous to those of this paper, but involving incomplete functions, may be found on starting from Liouville's '*formules générales,*' or from the elliptic theta equivalents of these was was done by Hermite* for $N_5(n)$ and $N_5(m, 5)$. He found

$$m = 8k + 5: \qquad N_5(m, 5) = G_1(n) + 2\Sigma G_1(m - 4a^2),$$

where $G_1(m)$ is defined by $G_1(m) = 4\Sigma(3d + d')$, the Σ extending to all positive integral solutions of $dd' = m$, $d' > 3d$. Incomplete functions have purposely been avoided in the sequel, as they appear to be less well adapted to computation than the primitives. In regard to Liouville's general formulas for which he did not publish proofs, and which, as indicated may be made to yield the results for 3, 5, $\cdots$, 13 squares, it was remarked by Hurwitz† that Stieltje's results possibly follow from some of them. It might seem, then, that all the simple functions should be derived directly from the same source, without the reference to elliptic functions implied by the use of Glaisher's results. But the reduction is only apparent, the origin of Liouville's formulas appearing most naturally in precisely those elliptic function identities that give the square theorems at once.

* Hermite, Oeuvres, IV, p. 238.

† Hurwitz, *Comptes Rendus*, 98 (1884), pp. 504–507. The two theorems which Liouville derives from his general formulas are insufficient for the proof of Stieltjes' results; cf. footnote to Section IV. Liouville did not indicate the connection between his theorems and representations as sums of 5 squares, nor did he carry out his intention of returning to the subject in a separate article.

II. General Formulas.

9. The notation has been explained in §§ 2, 5, and will be used as there given without further references. Summations being with respect to $\mu = 1, 3, 5, \cdots, \mu_1 = \pm 1, \pm 3, \pm 5, \cdots$, and $a = 1, 2, 3, \cdots, a_1 = 0, \pm 1, \pm 2, \pm 3, \cdots$, we have, in the usual notation of the elliptic theta constants,

$$\vartheta_2(q^4) = \Sigma q^{\mu_1^2} = 2\Sigma q^{\mu^2}, \tag{1}$$

$$\vartheta_3(q) = \Sigma q^{a_1^2} = 1 + 2\Sigma q^{a^2}. \tag{2}$$

Let $N'_r(n, s)$ denote the total number of representations of n as a sum of r squares, precisely s of which are odd and occupy the first s places in the representations; then, obviously

$$r!\, N'_r(n, s) = s!\,(r - s)!\, N_r(n, s), \tag{3}$$

$$\vartheta_2^s(q^4)\vartheta_3^{r-s}(q^4) = \sum_{n=1}^{\infty} q^n N'_r(n, s). \tag{4}$$

Consider the following identities, where $r > 1$:

$$\vartheta_2^{s+1}(q^4)\vartheta_3^{r-s-1}(q^4) = \vartheta_2(q^4) \times \vartheta_2^s(q^4)\vartheta_3^{r-s-1}(q^4), \tag{5}$$

$$\vartheta_2^s(q^4)\vartheta_3^{r-s}(q^4) = \vartheta_3(q^4) \times \vartheta_2^s(q^4)\vartheta_3^{r-s-1}(q^4), \tag{6}$$

$$\vartheta_2(q^4)\vartheta_3^{r-1}(q^4) = \vartheta_2(q^4) \times \vartheta_3^{r-1}(q^4), \tag{7}$$

$$\vartheta_3^r(q) = \vartheta_3(q) \times \vartheta_3^{r-1}(q). \tag{8}$$

By (4) the coefficient of q^n in the left member of (5) is $N'_r(n, s + 1)$. On using the second form of (1) for $\vartheta_2(q^4)$ on the right of (5), applying (4) to the second factor, and multiplying the resulting series (which are absolutely convergent), we find $2\Sigma N'_{r-1}(n - \mu^2, s)$ as the coefficient of q^n. Treating (6), (7), (8) similarly, equating coefficients of like powers of q, and using (3) to replace N' by its equivalent N, we find in this way from (5)–(8) the four fundamental identities, $r > 1$:

$$\text{(I)} \qquad (s + 1)N_r(n, s + 1) = 2r\Sigma N_{r-1}(n - \mu^2, s),$$

$$\text{(II)} \qquad (r - s)N_r(n, s) = r[N_{r-1}(n, s) + 2\Sigma N_{r-1}(n - 4a^2, s)],$$

$$\text{(III)} \qquad m = 4k + 1:\ N_r(m, 1) = 2r\left[\epsilon(m) + \Sigma N_{r-1}\left(\frac{m - \mu^2}{4}\right)\right],$$

$$\text{(IV)} \qquad N_r(n) = 2\epsilon(n) + N_{r-1}(n) + 2\Sigma N^*_{r-1}(n - a^2).$$

Again, from the definitions, $\vartheta_3^r(q^4) = \Sigma q^{4n} N_r(4n, 0)$; whence, on changing q into $\sqrt[4]{q}$,

$$\vartheta_3^r(q) = \Sigma q^n N_r(4n, 0), = \Sigma q^n N_r(n),$$

the last by (2) and the definition of $N_r(n)$ in § 5. Hence

$$(\mathrm{V}) \qquad N_r(4n, 0) = N_r(n).$$

10. It follows from (I). since $\mu^2 \equiv 1 \bmod 8$, that if $N_{r-1}(n, s)$ is primitive for $n \equiv g \bmod 8$, then $N_r(n, s+1)$ is simple when $n \equiv g+1 \bmod 8$. Similarly from (II), if $N_{r-1}(n, s)$ is primitive for $n \equiv g \bmod 4$, then $N_r(n, s)$ is simple for $n \equiv g \bmod 4$; and from (III), for $m \equiv 1 \bmod 4$, $N_r(m, 1)$ is simple if $N_{r-1}(n)$ is primitive, since $(m - \mu^2)/4$ may take any odd or even value, viz., it may take the values n; and the case (IV) shows that $N_r(n)$ is simple if $N_{r-1}(n)$ is primitive. Using (V) we get at once from (III), (IV) the corresponding forms for $N_r(m, 1)$ in terms of $N_{r-1}(m - \mu^2, 0)$, and of $N_r(4n, 0)$. Hence in sifting out the simples for a given r, we examine for what linear forms of n modulo 4 or 8, $N_{r-1}(n)$ is primitive, and apply the appropriate formulas of (I)–(V).

11. From (I), (II) we get, the c's being arbitrary constants:

$$(\mathrm{VI}) \qquad \sum_i c_i(1+s_i)N_r(n, 1+s_i) = 2r\sum_i c_i[\Sigma N_{r-1}(n-\mu^2, s_i)],$$

$$(\mathrm{VII}) \qquad \sum_i c_i(r-s_i)N_r(n, s_i) = r\sum_i c_i[N_{r-1}(n, s_i) + 2\Sigma N_{r-1}(n-4a^2, s_i)].$$

It is a remarkable fact, first stated by Liouville,* that for r odd and $n \equiv 0$ or $2 \bmod 4$, there always exist integers c_i, s_i depending upon r but not upon n such that $\Sigma_i c_i N_{r-1}(n, s_i)$ is primitive. Moreover, the s_i are in arithmetical progression. Suitably choosing the c_i, s_i we may therefore find in certain cases linear functions, viz., the left members of (VI), (VII) that are simple functions of n, although in general the individual $N_r(n, 1+s_i)$, $N_r(n, s_i)$ in the linear functions are neither primitive nor simple. To avoid reproducing the proofs of Liouville's theorems we shall write down the few necessary cases of (VI), (VII) directly from Glaisher's lists, whence they may be found by inspection.

12. The formulas (I)–(IV) may be reversed. In (I) change r into $r+1$, n into $n+1$, and solve for $N_r(n, s)$; in (II) replace r by $r+1$, and solve for $N_r(n, s)$; in (III) put $r+1$ for r, $4n+1$ for m, and solve for $N_r(n)$; and in (IV) change r into $r+1$ and solve for $N_r(n)$; then

$$(\mathrm{I}') \qquad 2(r+1)N_r(n, s) = (s+1)N_{r+1}(n+1, s+1) - 2(r+1)\Sigma N_r(n-8t, s),$$

$$(\mathrm{II}') \qquad (r+1)N_r(n, s) = (r+1-s)N_{r+1}(n, s) - 2(r+1)\Sigma N_r(n-4a^2, s),$$

$$(\mathrm{III}') \qquad 2(r+1)N_r(n) = N_{r+1}(4n+1, 1) - 2(r+1)[\epsilon(4n+1) + \Sigma N_r(n-2t)],$$

$$(\mathrm{IV}') \qquad N_r(n) = N_{r+1}(n) - 2\epsilon(n) - 2\Sigma N_r(n-a^2).$$

* *Journal des Math.*, (2) 6 (1861), 2 papers, pp. 233–, 369–. The proofs were not given, nor a method for determining the c_i when r is given. Both were published in *Bull. Amer. Math. Soc.*, Oct., 1919.

Of these, (IV′) shows that if $N_{r+1}(n)$ is primitive, then the values of $N_r(n)$ may be calculated by recurrence from $N_r(n-1)$, $N_r(n-4)$, $N_r(n-9)$, $\cdots$. Similarly for (I′)–(III′); remembering that t always denotes a triangular number. In each case the value of a primitive for one value of the variable has to be calculated in addition to the N_r functions; thus in (IV′) the assumed primitive is $N_{r+1}(n)$.

III. THREE SQUARES.

13. We first take the census (§ 5) for 3 squares:

$$N_3(4k) = N_3(4k, 0); \quad N_3(4k+1) = N_3(4k+1, 1);$$

$$N_3(4k+2) = N_3(4k+2, 2); \; N_3(8k+3) = N_3(8k+3, 3); \; N_3(8k+7) = 0;$$

next listing the known theorems for 2 squares:

$$(1) \quad N_2(n) = 4\xi(n); \qquad (2) \quad N_2(m) = N_2(m, 1) = 4\xi(m);$$

$$(3) \quad N_2(2m) = N_2(2m, 2) = 4\xi(m); \qquad \xi(2^a m) = \xi(m),$$

the last from the definitions in § 3. Having taken the census and tabulated the primitives for a given r, as in (1), (2), (3) above, we then consider (I)–(VII) of § 9, or so many of them as may be relevant to the particular r, here 3, and substitute successively $s = 1, 2, 3, \cdots$ until the total number of possible squares is exhausted, examining at each step the legitimate forms for n or m in $N_r(n, s+1)$, $N_r(n, s)$, etc., effecting this by an inspection of the census; and last, by referring to the primitives, such as (1)–(3) above, find the proper form for the right hand member of (I), (II), $\cdots$. Thus, putting $r = 3$, $s = 1$ in § 9 (I), the left becomes $3N_3(n, 2)$; and since by the census $N_3(n, 2)$ exists only when $n = 2m$, we must have on the right terms of the form $N_2(2m - \mu^2, 1)$, whose value, by (2) above, is $4\xi(2m - \mu^2)$, the variable $2m - \mu^2$ being odd and of the form $4k + 1$ as required by (2). Proceeding thus with all of the formulas (I)–(VII), we find the following cases, in which, *as always henceforth*, the integer in the numbering (2.1), (3.1), etc., of the formulas indicates from which of the primitive representations it has been derived, and the first decimal the particular one of (I)–(VII) used in the derivation. The rest of the numbering is self-explanatory.

Case I. $\qquad (s+1)N_3(n, s+1) = 6\Sigma N_2(n - \mu^2, s).$

$$(2.1) \qquad N_3(2m) = 12\Sigma\xi(2m - \mu^2);$$

$$(3.1) \qquad m = 8k+3: \quad N_3(m) = 8\Sigma\xi\left(\frac{m - \mu^2}{2}\right).$$

Case II. $(3-s)N_3(n, s) = 3[N_2(n, s) + 2\Sigma N_2(n-4a^2, s)]$.

Whence, for $s = 1, 2$:

$$(2.2) \qquad m = 4k+1: \quad N_3(m) = 6[\xi(m) + 2\Sigma\xi(m-4a^2)];$$

$$(3.2) \qquad N_3(2m) = 12[\xi(m) + 2\Sigma\xi(m-2a^2)].$$

The special form $4k+1$ of m restricts only the formula (2.2) with which it is written, having no relation to subsequent formulas; and so in all similar cases.

Case III. $m = 4k+1$: $N_3(m, 1) = 6\left[\epsilon(m) + \Sigma N_2\left(\frac{m-\mu^2}{4}\right)\right]$.

$$(1.3) \qquad m = 4k+1: \; N_3(m) = 6\left[\epsilon(m) + 4\Sigma\xi\left(\frac{m-\mu^2}{4}\right)\right].$$

Case IV. $\quad N_3(n) = 2\epsilon(n) + N_2(n) + 2\Sigma N_2(n-a^2)$.

$$(1.4) \qquad N_3(n) = 2[\epsilon(n) + 2\xi(n) + 4\Sigma\xi(n-a^2)].$$

Case V. $\quad N_3(4n) = 2[\epsilon(n) + 2\xi(n) + 4\Sigma\xi(n-a^2)]$.

Changing n into $2n$, applying the identity in § 2 in the form

$$\Sigma\xi(2n-a^2) = \Sigma\xi(2n-\mu^2) + \Sigma\xi(2n-4a^2),$$

and noting that $\xi(2b) = \xi(b)$, we get, as an alternative to the special case of (1.4) in which n is a multiple of 8;

$$(4) \qquad N_3(8n) = 2[\epsilon(2n) + 2\xi(n) + 4\Sigma\xi(2n-\mu^2) + 4\Sigma\xi(n-2a^2)];$$

and for $n = m$,

$$(5) \quad N_3(4m) = 2\left[\epsilon(m) + 2\xi(m) + 4\Sigma\xi\left(\frac{m-\mu^2}{2}\right) + 4\Sigma\xi(m-4a^2)\right];$$

both of which illustrate the way in which recurrences for the primitives may be written down on comparing with the equivalent forms deduced from the general (1.4).

IV. FIVE SQUARES.*

14. The census for 5 squares is

$N_5(4k) = N_5(4k, 0) + N_5(4k, 4)$; $\qquad N_5(8k+1) = N_5(8k+1, 1)$;

$N_5(2m) = N_5(2m, 2)$; $\qquad N_5(4k+3) = N_5(4k+3, 3)$;

$N_5(8k+5) \doteq N_5(8k+5, 1) + N_5(8k+5, 5)$.

Write $\lambda_r(n) = [2(-1)^n + 1]\zeta'_r(n)$, whence

$$\lambda_1(2n) = 3\zeta'_1(n), \; \lambda_1(m) = -\zeta_1(m);$$

* From the formulas (2.1), (6.22), (1.1), (7.2) of cases I, II we get Liouville's results (*J. des Math.*, (2), 4, p. 8); but not conversely.

then the primitive cases for 4 squares as given by Glaisher are:

(1) $N_4(2m, 2) = 24\zeta_1(m)$; (2) $N_4(4m, 4) = 16\zeta_1(m)$;

(3) $N_4(4n, 0) = 8(-1)^n\lambda_1(n) = N_4(n)$; (4) $N_4(2^a m) = 24\zeta_1(m)$;

(5) $N_4(m) = 8\zeta_1(m)$; (6) $m = 4k+1$, $N_4(m) = N_4(m, 1)$;

(7) $m = 4k+3$, $N_4(m) = N_4(m, 3)$.

Case I. $(s+1)N_5(n, s+1) = 10\Sigma N_4(n - \mu^2, s)$.

whence, for $s = 1, 2, 3, 4$:

(6.1) $$N_5(2m) = 40\Sigma\zeta_1(2m - \mu^2);$$

(1.1) $$m = 4k + 3: \quad N_5(m) = 80\Sigma\zeta_1\left(\frac{m - \mu^2}{2}\right);$$

(7.1) $$N_5(4n, 4) = 20\Sigma\zeta_1(4n - \mu^2);$$

(2.1) $$m = 8k + 5: \quad N_5(m, 5) = 32\Sigma\zeta_1\left(\frac{m - \mu^2}{4}\right).$$

The last was stated by Stieltjes, cf. § 7.

Case II. $(5 - s)N_5(n, s) = 5[N_4(n, s) + 2\Sigma N_4(n - 4a^2, s)]$.

Putting $s = 1, 2, 3, 4$, we get:

(6.21) $$m = 8k + 1: \quad N_5(m) = 10[\zeta_1(m) + 2\Sigma\zeta_1(m - 4a^2)];$$

(6.22) $$m = 8k + 5: \quad N_5(m, 1) = 10[\zeta_1(m) + 2\Sigma\zeta_1(m - 4a^2)];$$

(1.2) $$N_5(2m) = 40[\zeta_1(m) + 2\Sigma\zeta_1(m - 2a^2)];$$

(7.2) $$m + 4k + 3: \quad N_5(m) = 20[\zeta_1(m) + 2\Sigma\zeta_1(m - 4a^2)];$$

(2.21) $$N_5(4m, 4) = 80[\zeta_1(m) + 2\Sigma\zeta_1(m - 4a^2)];$$

(2.22) $$N_5(8n, 4) = 160\Sigma\zeta_1(2n - \mu^2).$$

Case III. $m = 4k + 1$: $N_5(m, 1) = 10\left[\epsilon(m) + \Sigma N_4\left(\frac{m - \mu^2}{4}\right)\right]$.

(3.31) $$m = 8k + 1: \quad N_5(m) = 10\left[\epsilon(m) + 24\Sigma\zeta_1'\left(\frac{m - \mu^2}{8}\right)\right],$$

(3.32) $$m = 8k + 5: \quad N_5(m, 1) = 80\Sigma\zeta_1\left(\frac{m - \mu^2}{2}\right).$$

Cases IV, V. $N_4(4n, 0) = 2\epsilon(n) + N_4(4n, 0) + 2\Sigma N_4(4n - 4a^2, 0)$.

(3.4) $N_5(n) = N_5(4n, 0) = 2\epsilon(n) + 8(-1)^n[\lambda_1(n) + 2\Sigma(-1)^a\lambda_1(n - a^2)]$.

15. From the census, $N_5(4k) = N_5(4k, 0) + N_5(4k, 4)$; hence from

(7.1), (3.4) we get an alternative form of $N_5(4n)$ which may be compared with that given by (3.4). Similarly combining the formulas (2. 1), (3.32), we have

$$(8) \qquad m = 8k + 5: \ N_5(m) = 112\Sigma\zeta_1\left(\frac{m - \mu^2}{4}\right).$$

V. SEVEN SQUARES.

16. The census is

$$N_7(4k) = N_7(4k, 0) + N_7(4k, 4);$$

$$N_7(2m) = N_7(2m, 2) + N_7(2m, 6);$$

$$N_7(8k + 7) = N_7(8k + 7, 3) + N_7(8k + 7, 7);$$

$$N_7(4k + 1) = N_7(4k + 1, 1) + N_7(4k + 1, 5);$$

$$N_7(8k + 3) = N_7(8k + 3, 3);$$

also

$$\xi'_r(m) = (-1\,|\,m)\xi_r(m),$$

and from Glaisher's lists the primitives for 6 squares are:

(1) $N_6(2m, 2) = 60\xi'_2(m)$; (2) $m = 4k + 3 : N_6(m, 3) = -20\xi_2(m)$;

(3) $N_6(4n, 4) = 240\xi'_2(n)$; (4) $m = 4k + 3 : N_6(2m, 6) = -8\xi_2(m)$;

$$(5) \ N_6(n) = N_6(4n, 0) = 4[4\xi'_2(n) - \xi_2(n)].$$

Case I. $(s + 1)N_7(n, s + 1) = 14\Sigma N_6(n - \mu^2, s)$.

Putting $s = 2, 3, 4, 6$, we find:

$$(1.1) \qquad m = 4k + 3: \ N_7(m, 3) = 280\Sigma\xi'_2\left(\frac{m - \mu^2}{2}\right);$$

$$(1.11) \qquad m = 8k + 3: \ N_7(m) = 280\Sigma\xi_2\left(\frac{m - \mu^2}{2}\right);$$

$$(1.12) \qquad m + 8k + 7: \ N_7(m, 3) = -280\Sigma\xi_2\left(\frac{m - \mu^2}{2}\right);$$

$$(2.1) \qquad N_7(4n, 4) = -70\Sigma\xi_2(4n - \mu^2);$$

$$(3.1) \qquad m = 4k + 1: \ N_7(m, 5) = 672\Sigma\xi'_2\left(\frac{m - \mu^2}{4}\right);$$

$$(4.1) \qquad m = 8k + 7: \ N_7(m, 7) = -16\Sigma\xi_2\left(\frac{m - \mu^2}{2}\right).$$

From (1.12), (4.1) we see that for $m \equiv 7 \bmod 8$, $2N_7(m, 3) = 35N_7(m, 7)$. Many such relations may be read off from the lists for 3, $\cdots$, 13 squares.

Case II. $(7-s)N_7(n, s) = 7[N_6(n, s) + 2\Sigma N_6(n-4a^2, s)]$. Putting $s = 2, 3, 4, 6$:

(1.2) $$N_7(2m, 2) = 84[\xi_2'(m) + 2\Sigma\xi_2'(m-2a^2)];$$

(2.21) $$m = 8k+3:\quad N_7(m) = -35[\xi_2(m) + 2\Sigma\xi_2(m-4a^2)];$$

(2.22) $$m = 8k+7:\quad N_7(m, 3) = -35[\xi_2(m) + 2\Sigma\xi_2(m-4a^2)];$$

(3.2) $$N_7(4n, 4) = 560[\xi_2'(n) + 2\Sigma\xi_2'(n-a^2)];$$

(4.2) $$m = 4k+3:\quad N_7(2m, 6) = -56[\xi_2(m) + 2\Sigma\xi_2(m-8a^2)].$$

Case III. $$m = 4k+1:\quad N_7(m, 1) = 14\left[\epsilon(m) + \Sigma N_6\left(\frac{m-\mu^2}{2}\right)\right].$$

(5.3) $$m=4k+1:\quad N_7(m, 1) = 14\left[\epsilon(m)+4\Sigma\left\{4\xi_2'\left(\frac{m-\mu^2}{4}\right)-\xi_2\left(\frac{m-\mu^2}{4}\right)\right\}\right].$$

Cases IV, V. $$N_7(4n, 0) = N_7(n) =$$

(5.4) $$2\epsilon(n) + 4[4\xi_2'(n) - \xi_2(n)] + 8\Sigma[4\xi_2'(n-a^2) - \xi_2(n-a^2)].$$

The equivalent of (5.4) was stated by Stieltjes, *C.R.*, 31 Dec., 1884.

Cases VI, VII. For the first time these enter. From Glaisher's theorems for 6 squares (loc. cit., p. 10), we find on eliminating non-primitives,

(7) $$m = 4k+1:\quad N_6(m, 1) + N_6(m, 5) = 12\xi_2(m).$$

Hence, taking § 11 (VI) in the form

$$(1+s_1)N_7(n, 1+s_1) + (1+s_2)N_7(n, 1+s_2)$$
$$= 14\Sigma[N_6(n-\mu^2, s_1) + N_6(n-\mu^2, s_2)],$$

and choosing $(s_1, s_2) = (1, 5)$, we get, on referring to the census,

(7.6) $$m = 4k+1:\quad N_7(2m, 2) + 3N_7(2m, 6) = 84\Sigma\xi_2(2m-\mu^2);$$

and from (VII) in the same way,

(7.7) $$m = 4k+1:\quad 3N_7(m, 1) + N_7(m, 5) = 42[\xi_2(m) + 2\Sigma\xi_2(m-4a^2)].$$

Combining (7.7), (3.1) or (7.7), (5, 3) we find alternative forms for $N_7(m, 1)$ or $N_7(m, 5)$ ($m \equiv 1 \bmod 4$) respectively; and from (7.6), (1.2):

(8) $$m = 4k+1:\quad N_7(2m, 6) = 28[\Sigma\xi_2(2m-\mu^2) - \xi_2(m) - 2\Sigma\xi_2'(m-2a^2)],$$

which is at once expressible in terms of ξ_2' alone on using the identity of § 2.

17. From the lists in § 16 we write down the following additional complete formulas by reference to the census. Some, such as $N_7(4n)$, which are not much better adapted to computation than those given directly by (5.4), have been omitted.

(9) $m=4k+1$: $N_7(m) = 14\left[\epsilon(m)+64\Sigma\xi_2'\left(\frac{m-\mu^2}{4}\right)-4\Sigma\xi_2\left(\frac{m-\mu^2}{4}\right)\right]$;

(10) $m=4k+1$: $N_7(2m) = 28[\xi_2(m)+\Sigma\xi_2(2m-\mu^2)+4\Sigma\xi_2'(m-2a^2)]$;

(11) $m=4k+3$: $N_7(2m) = -28[5\xi_2(m)-6\Sigma\xi_2(m-2\mu^2)$
$+10\Sigma\xi_2(m-8a^2)]$;

(12) $m=8k+7$: $N_7(m) = -296\Sigma\xi_2\left(\frac{m-\mu^2}{2}\right)$.

VI. NINE SQUARES.

18. The census is

$$N_9(4n) = N_9(4n, 0) + N_9(4n, 4) + N_9(4n, 8),$$
$$m = 8k+1: \ N_9(m) = N_9(m, 1) + N_9(m, 5) + N_9(m, 9),$$
$$N_9(2m) = N_9(2m, 2) + N_9(2m, 6),$$
$$m = 8k+5: \ N_9(m) = N_9(m, 1) + N_9(m, 5),$$
$$m = 4k+3: \ N_9(m) = N_9(m, 3) + N_9(m, 7).$$

To state the primitives for 8 squares it is convenient to introduce $\rho_3(n)$, $\alpha_3(n)$, where $\rho_r(n) = \zeta_r'(n) - \zeta_r''(n)$, and $\alpha_r(n) = n^r\zeta_{-r}'(n) =$ the sum of the rth powers of all those divisors of n whose conjugates are odd. From these definitions we have at once the useful identities:

$$\alpha_3(m) = \zeta_3(m), \qquad \alpha_3(2m) = 8\zeta_3(m), \qquad \alpha_3(2n) = 8\alpha_3(n),$$
$$\rho_3(m) = \zeta_3(m), \qquad \rho_3(2m) = -7\zeta_3(m), \qquad \rho_3(2n) = 8\rho_3(n) - 15\zeta_3'(n),$$
$$7\rho_3(n) + 8\alpha_3(n) = 15\zeta_3'(n).$$

The primitives for 8 squares, from Glaisher, are:

(1) $N_8(4n, 4) = 1120\alpha_3(n)$; (2) $N_8(8n, 8) = 256\alpha_3(n)$;

(3) $N_8(n) = -16(-1)^n\rho_3(n)$; (4) $N_8(4n, 0) = N_8(n)$.

Case I. $(s+1)N_9(n, s+1) = 18\Sigma N_8(n-\mu^2, s)$.

Whence, for $s = 4, 8$:

(1.1) $m = 4k+1$: $N_9(m, 5) = 4032\Sigma\alpha_3\left(\frac{m-\mu^2}{4}\right)$;

(1.11) $m = 8k+1$: $N_9(m, 5) = 32256\Sigma\alpha_3\left(\frac{m-\mu^2}{8}\right)$;

(1.12) $m = 8k+5$: $N_9(m, 5) = 4032\Sigma\zeta_3\left(\frac{m-\mu^2}{4}\right)$;

(2.1) $m = 8k+1$: $N_9(m, 9) = 512\Sigma\alpha_3\left(\frac{m-\mu^2}{8}\right)$.

Case II. $(9 - s)N_9(n, s) = 9[N_8(n, s) + 2\Sigma N_8(n - 4a^2, s)]$.

Putting $s = 4, 8$, and separating the cases according as $m \equiv 0$ or $4 \bmod 8$, we find, after some short reductions:

(1.2) $N_9(4n, 4) = 2016[\alpha_3(n) + 2\Sigma\alpha_3(n - a^2)]$;

(1.21) $N_9(4m, 4) = 2016\left[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2) + 16\Sigma\alpha_3\left(\frac{m - \mu^2}{2}\right)\right]$;

(1.22) $N_9(8n, 4) = 4032[4\alpha_3(n) + 8\Sigma\alpha_3(n - 2a^2) + \Sigma\zeta_3(2n - \mu^2)]$;

(2.21) $N_9(4m, 8) = 4608\Sigma\alpha_3\left(\frac{m - \mu^2}{2}\right)$;

(2.22) $N_9(8n, 8) = 2304[\alpha_3(n) + 2\Sigma\alpha_3(n - 2a^2)]$.

Case III. $m = 4k + 1$: $N_9(m, 1) = 18\left[\epsilon(m) + \Sigma N_8\left(\frac{m - \mu^2}{4}\right)\right]$.

(3.31) $m = 8k + 1$: $N_9(m, 1)$

$$= 18\left[\epsilon(m) + 240\Sigma\zeta_3'\left(\frac{m - \mu^2}{8}\right) - 128\Sigma\rho_3\left(\frac{m - \mu^2}{8}\right)\right];$$

(3.32) $m = 8k + 5$: $N_9(m, 1) = 288\Sigma\zeta_3\left(\frac{m - \mu^2}{4}\right)$.

Cases IV, V. $N_9(n) = N_9(4n, 0) =$

(3.4) $2\epsilon(n) - 16(-1)^n[\rho_3(n) + 2\Sigma(-1)^a\rho_3(n - a^2)]$.

Cases VI, VII. From Glaisher's results for 8 squares we have:

(5) $m = 4k + 1$: $N_8(m, 1) + N_8(m, 5) = 16\zeta_3(m)$;

(6) $N_8(2m, 2) + N_8(2m, 6) = 112\zeta_3(m)$;

(7) $m = 4k + 3$: $N_8(m, 3) + N_8(m, 7) = 16\zeta_3(m)$.

Hence, putting $(s_1, s_2) = (1, 5), (2, 6), (3, 7)$ in (VI), (VII) of § 11, and comparing as usual with the census and primitives:

(5.6) $N_9(2m, 2) + 3N_9(2m, 6) = 144\Sigma\zeta_3(2m - \mu^2)$;

(6.6) $m = 4k + 3$: $3N_9(m, 3) + 7N_9(m, 7) = 2016\Sigma\zeta_3\left(\frac{m - \mu^2}{2}\right)$;

(7.6) $N_9(4n, 4) + 2N_9(4n, 8) = 72\Sigma\zeta_3(4n - \mu^2)$;

(5.7) $m = 4k + 1$: $2N_9(m, 1) + N_9(m, 5) = 36[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2)]$.

(6.7) $$7N_9(2m, 2) + 3N_9(2m, 6) = 1008[\zeta_3(m) + 2\Sigma\zeta_3(m - 2a^2)];$$

(7.7) $$m = 4k + 3: \ 3N_9(m, 3) + N_9(m, 7) = 72[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2)].$$

Solving these we get:

(9) $$N_9(2m, 2) = 24[7\zeta_3(m) + 14\Sigma\zeta_3(m - 2a^2) - \Sigma\zeta_3(2m - \mu^2)];$$

(10) $$N_9(2m, 6) = 56[-\zeta_3(m) - 2\Sigma\zeta_3(m - 2a^2) + \Sigma\zeta_3(2m - \mu^2)];$$

(11) $m = 4k + 3$:
$$N_9(m, 3) = 28\left[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2) - 4\Sigma\zeta_3\left(\frac{m - \mu^2}{2}\right)\right];$$

(12) $m = 4k + 3$:
$$N_9(m, 7) = 12\left[-\zeta_3(m) - 2\Sigma\zeta_3(m - 4a^2) + 28\Sigma\zeta_3\left(\frac{m - \mu^2}{2}\right)\right].$$

Similarly from (1.1), (5.7):

(13) $m = 4k + 1$:
$$N_9(m, 1) = 18\left[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2) - 112\Sigma\alpha_3\left(\frac{m - \mu^2}{4}\right)\right].$$

19. Combining certain of the formulas in § 18 according to the census we find the additional complete cases in the following list. Thus, the value of $N_9(2m) = N_9(2m, 2) + N_9(2m, 6)$ follows from (9), (10). Some that may be found in this way have been omitted.

(14) $m = 8k + 1$:
$$N_9(m) = 18\zeta_3(m) + 36\Sigma\zeta_3(m - 4a^2) + 16640\Sigma\alpha_3\left(\frac{m - \mu^2}{8}\right);$$

(15) $m = 8k + 5$:
$$N_9(m) = 4320\Sigma\zeta_3\left(\frac{m - \mu^2}{4}\right);$$

(16) $m = 4k + 3$:
$$N_9(m) = 16\left[\zeta_3(m) + 2\Sigma\zeta_3(m - 4a^2) + 14\Sigma\zeta_3\left(\frac{m - \mu^2}{2}\right)\right];$$

(17) $$N_9(2m) = 16[7\zeta_3(m) + 14\Sigma\zeta_3(m - 2a^2) + 2\Sigma\zeta_3(2m - \mu^2)].$$

VII. ELEVEN SQUARES.*

20. The census is in § 5; and the primitives for ten squares were found by Glaisher to be:

* The references in Bachmann's Zahlentheorie, Bd. IV, and the German Encyclopaedia to Liouville's consideration of this case are misprints; Liouville published nothing concerning 11 squares.

(1) $5N_{10}(n) = 4[\xi_4(n) + 16\xi_4'(n)]$; (2) $N_{10}(2m, 2) = -36\xi_4(m)$;

(3) $N_{10}(4n, 4) = 2688\xi_4'(n)$; (4) $N_{10}(2m, 6) = -168\xi_4(m)$;

(5) $N_{10}(4n, 8) = 576\xi_4'(n)$; (6) $N_{10}(4n, 0) = N_{10}(n)$,

in all of which $n = 2^b m$, $m = 4k + 3$, $b \geqq 0$. For the same value of n, $\xi_4'(n) = -2^{4b}\xi_4(m)$, $\xi_4(n) = \xi_4(m)$; and the numbers $n - a^2$ ($a = 1, 2, 3, \cdots$) are all of the form $2^c(4k + 3)$, $c \geqq 0$, when and only when $b = 0, 2$, and $m = 8k + 7$. Hence we find only the following simple expressions for 11 squares.

Case I. $(s + 1)N_{11}(n, s + 1) = 22\Sigma N_{10}(n - \mu^2, s)$.

Whence, for $s = 2, 6$:

$$(2.1) \qquad m = 8k + 7\colon\ N_{11}(m, 3) = -264\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right);$$

$$(4.1) \qquad m = 8k + 7\colon\ N_{11}(m, 7) = -528\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right).$$

Case II. $(11 - s)N_{11}(n, s) = 11[N_{10}(n, s) + 2\Sigma N_{10}(n - 4a^2, s)]$.

Comparing this with (3), (5), we see that when and only when n is of either form $8k + 7$, $4(8k + 7)$,

$$(3.2) \qquad N_{11}(4n, 4) = 4224[\xi_4'(n) + 2\Sigma\xi_4'(n - a^2)];$$

$$(5.2) \qquad N_{11}(4n, 8) = 2112[\xi_4'(n) + 2\Sigma\xi_4'(n - a^2)].$$

Case IV. $N_{11}(n) = 2\epsilon(n) + N_{10}(n) + 2\Sigma N_{10}(n - a^2)$.

Hence for n as in case II, and $m = 8k + 7$:

$$(1.4) \qquad 5N_{11}(n) = 4[\xi_4(n) + 16\xi_4'(n) + 2\Sigma\{\xi_4(n - a^2) + 16\xi_4'(n - a^2)\}];$$

$$(1.41) \quad N_{11}(m) = -12\left[\xi_4(m) + 2\Sigma\xi_4(m - 4a^2) + 32\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right)\right].$$

21. Combining (2.1), (4.1) by the census,

$$(7) \qquad m = 8k + 7\colon\ N_{11}(m) = -792\Sigma\xi_4\left(\frac{m - \mu^2}{2}\right).$$

The census gives also $N_{11}(4m) = N_{11}(4m, 0) + N_{11}(4m, 4) + N_{11}(4m, 8)$, and (1.4) the value of $N_{11}(4m, 0)$; hence from these and (3.2), (5.2) may be derived recurrences for the ξ_4, ξ_4'; and similarly from (1.4) and (7).

22. By eliminating the non-primitives from the formulas for $N_{10}(n, r)$ as given by Glaisher, it is easy to deduce several relations of the forms (VI), (VII) of § 11, and therefore to find simple linear functions of N's. But, as in preceding sections, it is verified without difficulty that the com-

plete system of such relations when combined with the formulas of §§ 20, 21 is insufficient for the determination of any N_{11} not already listed. This verification, presenting nothing new, is omitted. The like may be shown to hold for 13, 15, 17, 19, 21, 23, 25 squares; and when we pass to 13 squares (the next case considered) there is the concurrent disappearance of primitives for the associated even number of squares. Whether both or either of these failures occurs always for m squares when > 25, has been shown in another paper to depend upon the divisors common to certain binomial coefficients and the numerators of the numerical coefficients in the power series for the Jacobian elliptic functions, and need only be mentioned here. We remark, however, that one aspect of the failure is permanent after 13 squares; viz., the number of linear relations between the several $N_m(n, r)$ is always less than the number of unknown functions.

VIII. THIRTEEN SQUARES.

23. It is unnecessary to take the census. $\alpha_r(n)$ is defined in § 18; and we now introduce $\beta_r(n)$, $= \zeta_r'(n) - \zeta_r''(n) + 2\alpha_r(n)$. This function is susceptible of many transformations; in particular it may be expressed in terms of 2^s and $\zeta_r(m)$, where $n = 2^s m$, but the form stated is sufficient for our purpose. The only primitives for 12 squares are

$$(1)\ N_{12}(2n) = 8\beta_5(2n); \qquad (2)\ N_{12}(8n, 4) = N_{12}(8n, 8) = 126720\alpha_5(n).$$

An equivalent form of (1) was stated with insufficient proof by Liouville,* and first proved by Glaisher, from whose paper both are transcribed. We shall consider the primitive $N_{12}(8n, 0)$ as included under (1). It is readily seen that Cases I, III alone are applicable.

Case I. $\qquad (s + 1)N_{13}(n, s + 1) = 26\Sigma N_{12}(n - \mu^2, s).$

Whence, for $s = 4, 8$:

$$(2.11) \qquad m = 8k + 1:\ N_{13}(m, 5) = 658944\Sigma\alpha_5\left(\frac{m - \mu^2}{8}\right);$$

$$(2.12) \qquad m = 8k + 1:\ N_{13}(m, 9) = 366080\Sigma\alpha_5\left(\frac{m - \mu^2}{8}\right).$$

Hence $5N_{13}(m, 5) = 9N_{13}(m, 9)$ when $m \equiv 1 \bmod 8$.

* *Journal des Math.*, (2) 6 (1861), p. 206. The proof is insufficient because it is made to depend upon a theorem which Liouville did not prove, and whose proof is more difficult than that of the special result for 12 squares; cf. Bachmann, Zahlentheorie, Bd. IV, p. 668. The same applies also to the primitives for 10 squares, first stated by Eisenstein and Liouville, but first proved by Glaisher.

Case III. $m = 4k + 1$: $N_{13}(m, 1) = 26\left[\epsilon(m) + \Sigma N_{12}\left(\frac{m - \mu^2}{4}\right)\right]$.

(1.3) $m = 8k + 1$: $N_{13}(m, 1) = 26\left[\epsilon(m) + 8\Sigma\beta_5\left(\frac{m - \mu^2}{4}\right)\right]$.

24. The census gives

$$m = 8k + 1:\ N_{13}(m) = N_{13}(m, 1) + N_{13}(m, 5) + N_{13}(m, 9).$$

Hence, from § 23, we have

(3) $m = 8k + 1$:

$$N_{13}(m) = 26\left[\epsilon(m) + 8\Sigma\beta_5\left(\frac{m - \mu^2}{4}\right) + 39424\Sigma\alpha_5\left(\frac{m - \mu^2}{8}\right)\right].$$

IX. RECURRENCES.

25. In the numbering (3.1), etc., of the following formulas the integer $= r$, and the first decimal $= s$, and the resulting recurrence is written down from the general formula at the head of its set by substituting the values of (r, s) thus defined. *E.g.*, we get the first (3.1) on putting $r = 3$ $s = 1$ in (I′), and referring to § 14 for the value of

$$(s + 1)N_{r+1}(m + 1, s + 1), = 2N_4(m + 1, 2),$$

noting that $m + 1$ is an even integer, say $2n$; hence $m = 2n - 1$, and therefore being any odd positive integer, is denoted by m in (3.1). Similarly all the recurrences under the several cases (I′)–(IV′) are written down by inspection on glancing first at the primitives for $N_{r+1}(m + 1, s + 1)$, $N_{r+1}(n, s)$, $N_{r+1}(4n + 1, 1)$, $N_{r+1}(n)$ respectively, and giving r the values 3, 5, 7, 9, 11, in succession. We recall that t represents always a triangular number. Cases I′–IV′ are from § 12.

Case I′. $2(r + 1)N_r(n, s) = (s + 1)N_{r+1}(n + 1, s + 1)$
$- 2(r + 1)\Sigma N_r(n - 8t, s)$.

(3.1) $m = 4k + 1$: $N_3(m) = 6\zeta_1\left(\frac{m + 1}{2}\right) - \Sigma N_3(m - 8t)$;

(3.2) $N_3(2m) = 3\zeta_1(2m + 1) - \Sigma N_3(2m - 8t)$;

(3.3) $m = 8k + 3$: $N_3(m) = 8\zeta_1\left(\frac{m + 1}{4}\right) - \Sigma N_3(m - 8t)$;

(5.11) $m = 8k + 1$: $N_5(m) = 10\zeta_2\left(\frac{m + 1}{2}\right) - \Sigma N_5(m - 8t)$;

(5.12) $m = 8k + 5$: $N_5(m, 1) = -10\xi_2\left(\frac{m + 1}{2}\right) - \Sigma N_5(m - 8t, 1)$;

(5.2) $$N_5(2m) = -5\xi_2(2m+1) - \Sigma N_5(2m-8t);$$

(5.3) $m = 4k+3$: $$N_5(m) = 80\xi_2'\left(\frac{m+1}{4}\right) - \Sigma N_5(m-8t);$$

(5.5) $m = 8k+5$: $$N_5(m, 5) = -4\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m-8t, 5);$$

(5.51) $m = 8k+5$: $$N_5(m) = -14\xi_2\left(\frac{m+1}{2}\right) - \Sigma N_5(m-8t);$$

(7.31) $m = 8k+3$: $$N_7(m) = 280\alpha_3\left(\frac{m+1}{4}\right) - \Sigma N_7(m-8t);$$

(7.32) $m = 8k+7$: $$N_7(m, 3) = 2240\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t, 3);$$

(7.7) $m = 8k+7$: $$N_7(m, 7) = 128\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t, 7);$$

(7.71) $m = 8k+7$: $$N_7(m) = 2368\alpha_3\left(\frac{m+1}{8}\right) - \Sigma N_7(m-8t);$$

(9.1) $m = 8k+5$: $$5N_9(m, 1) = -18\xi_4\left(\frac{m+1}{2}\right) - 5\Sigma N_9(m-8t, 1);$$

(9.31) $m = 16k+11$: $$5N_9(m, 3) = 2688\xi_4'\left(\frac{m+1}{4}\right) - 5\Sigma N_9(m-8t, 3);$$

(9.5) $m = 8k+5$: $$5N_9(m, 5) = -252\xi_4\left(\frac{m+1}{2}\right) - 5\Sigma N_9(m-8t, 5);$$

(9.51) $m = 8k+5$: $$N_9(m) = -54\xi_4\left(\frac{m+1}{2}\right) - \Sigma N_9(m-8t);$$

(9.7) $m = 16k+11$: $$5N_9(m, 7) = 1152\xi_4'\left(\frac{m+1}{4}\right) - 5\Sigma N_9(m-8t, 7);$$

(9.71) $m = 16k+11$: $$N_9(m) = 768\xi_4'\left(\frac{m+1}{4}\right) - \Sigma N_9(m-8t);$$

(11.3) $m = 8k+7$: $$N_{11}(m, 3) = 21120\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t, 3);$$

(11.7) $m = 8k+7$: $$N_{11}(m, 7) = 44240\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t, 7);$$

(11.71) $m = 8k+7$: $$N_{11}(m) = 63360\alpha_5\left(\frac{m+1}{8}\right) - \Sigma N_{11}(m-8t).$$

Case II'. $(r+1)N_r(n, s) = (r+1-s)N_{r+1}(n, s)$
$$- 2(r+1)\Sigma N_r(n - 4a^2, s).$$

(3.1) $m = 4k+1$: $N_3(m) = 6\zeta_1(m) - 2\Sigma N_3(m - 4a^2)$;

(3.2) $N_3(2m) = 12\zeta_1(m) - 2\Sigma N_3(2m - 4a^2)$;

(3.3) $m = 8k+3$: $N_3(m) = 2\zeta_1(m) - 2\Sigma N_3(m - 4a^2)$;

(5.2) $N_5(2m) = 40\xi_2'(m) - 2\Sigma N_5(2m - 4a^2)$;

(5.3) $m = 4k+3$: $N_5(m) = -10\xi_2(m) - 2\Sigma N_5(m - 4a^2)$;

(5.4) $N_5(4n, 4) = 80\xi_2'(n) - 2\Sigma N_5(4n - 4a^2, 4)$;

(7.4) $N_7(4n, 4) = 560\alpha_3(n) - 2\Sigma N_7(4n - 4a^2, 4)$;

(9.2) $m = 4k+3$: $5N_9(2m, 2) = -144\xi_4(m) - 10\Sigma N_9(2m - 4a^2, 2)$;

(9.4) $n = 2^b(4k+3)$, $b \geqq 0$:
$$5N_9(4n, 4) = 8064\xi_4'(n) - 10\Sigma N_9(4n - 4a^2, 4);$$

(9.6) $m = 4k+3$: $5N_9(2m, 6) = -336\xi_4(m) - 10\Sigma N_9(2m - 4a^2, 6)$;

(9.61) $m = 4k+3$: $N_9(2m) = -96\xi_4(m) - 10\Sigma N_9(2m - 4a^2)$;

(9.8) $n = 2^b(4k+3)$, $b \geqq 0$:
$$5N_9(4n, 8) = 576\xi_4'(n) - 10\Sigma N_9(4n - 4a^2, 8);$$

(11.4) $N_{11}(8n, 4) = 84480\alpha_5(n) - 2\Sigma N_{11}(8n - 4a^2, 4)$;

(11.8) $N_{11}(8n, 8) = 42240\alpha_5(n) - 2\Sigma N_{11}(8n - 4a^2, 8)$.

Case III'. $2(r+1)N_r(n) = N_{r+1}(4n+1, 1)$
$$- 2(r+1)[\epsilon(4n+1) + \Sigma N_r(n - 2t)].$$

(3.1) $N_3(n) = \zeta_1(4n+1) - \epsilon(4n+1) - \Sigma N_3(n - 2t)$.

Case IV'. $N_r(n) = N_{r+1}(n) - 2\epsilon(n) - 2\Sigma N_r(n - a^2)$.

(3) $N_3(n) = 8(-1)^n\lambda_1(n) - 2\epsilon(n) - 2\Sigma N_3(n - a^2)$;

(5) $N_5(n) = 4[\xi_2'(n) - \xi_2(n)] - 2\epsilon(n) - 2\Sigma N_5(n - a^2)$;

(7) $N_7(n) = -16(-1)^n\rho_3(n) - 2\epsilon(n) - 2\Sigma N_7(n - a^2)$;

(9) $n = 2^b(4k+3)$, $b \geqq 0$:
$$5N_9(n) = 4[\xi_4(n) + 16\xi_4'(n)] - 10\epsilon(n) - 10\Sigma N_9(n - a^2);$$

(11) $N_{11}(2n) = 8\beta_5(n) - 2\epsilon(2n) - 2\Sigma N_{11}(2n - a^2)$.

Since $n - a^2$ is not always of the form $2^b(4k+3)$, (9) does not enable us

to calculate the values of $N_9(n)$ by recurrence alone; and the same holds for (11), half the values of $2n - a^2$ being odd. The like applies to certain of the formulas under I′–III′; but in conjunction with the results of sections III–VII, even the incomplete recurrences are a material aid to computation.

26. Comparing the finite sums given by $N_r(n)$ under cases IV in sections III–VII with those determined under cases I–III, we may derive useful recurrences for certain of the primitives. These, which are of the same general type as the two examples given by Liouville (cf. § 14, footnote), we omit.

UNIVERSITY OF WASHINGTON.

ON A CERTAIN CLASS OF RATIONAL RULED SURFACES.

By Arnold Emch.

1. Introduction.

As is well known, ruled surfaces may be generated or defined in a number of ways. There exists, for example, a one-to-one correspondence between ruled surfaces, or scrolls as Cayley calls them, and a certain class of partial differential equations, so that the theories of the two classes are abstractly equivalent.

A much favored method, especially in descriptive geometry, consists in considering ruled surfaces as continuous sets of straight lines (generatrices), which intersect three fixed curves, (the directrices), simultaneously. If these are algebraic curves of orders l, m, n, with no points in common, the ruled surface which they define is, in general, of order $2lmn$.

Frequently ruled surfaces are also defined as systems of elements, either common to two rectilinear congruences, or to three rectilinear complexes.

Of great importance for the following investigation is the definition of ruled surfaces as systems of lines which join corresponding points of an (α, β)-correspondence between the points of two algebraic curves C_m and C_n of orders m and n. If the curves are plane, and if to a point of C_m correspond α points of C_n, and to a point of C_n β points of C_m, then the order of the surface is, in general, $\alpha m + \beta n$.

Finally there is the cinematic method in which ruled surfaces are generated by the continuous movement of the generatrix according to some definite cinematical law. In this connection the description of the hyperboloid of rotation of one sheet by a straight line rotating about a fixed axis is well known. The literature seems to contain but little about this method of generating ruled surfaces.* A number of treatises on differ-

In this paper the results of an investigation of a rather extended class of ruled surfaces are presented, which are defined cinematically. The class is

* E. M. Blake, "Two Plane Movements Generating Quartic Scrolls," *Transactions of the American Mathematical Society*, Vol. 1, pp. 421–429 (1900).

ential geometry contain chapters on cinematically generated surfaces.†

† Darboux, "Leçons sur la théorie générale des surfaces," Vol. 1, 2d ed., pp. 127–150 (1914).

Eisenhart, "A Treatise on the Differential Geometry of Curves and Surfaces," pp. 146–148 (1909).

obtained as follows: Given a directrix circle C_2 and a directrix line C_1, which passes through the center of C_2 and is at right angles to the plane of C_2. The generatrix g moves so that a fixed point M of g moves along C_2 uniformly, while g in every position passes through C_1. The plane e through C_1 in which g lies evidently rotates about C_1 with the same velocity $k\theta$ as M. In this plane e, the generatrix g rotates at the same time with uniform velocity $\mu k\theta$ about M. It will be shown that g generates a rational algebraic surface of the class when $\mu = p/q$ is a rational fraction, and that this class thus described is equivalent to a class of ruled surfaces obtained by means of an (α, β)-correspondence between C_1 and C_2, in which α and β will be determined hereafter.

Among the most important references bearing more closely upon the subject may be mentioned papers by Cremona,* Armamente,† Cayley,‡ Schwarz,§ Noether,|| Clebsch,¶ Picard.**

2. PARAMETRIC EQUATIONS AND ORDER OF THE SURFACE.

Let C_1 coincide with the z-axis, so that C_2 lies in the xy-plane, and g any position of the generatrix, P any point on g, P' its projection upon the xy-plane, and M the intersection of g with C_2. By ρ denote the radius vector OP', whose direction forms an angle θ with the positive part of the x-axis. In the plane e through C_1, in which g lies, g is determined by the angle ψ which g makes with the positive direction of the perpendicular through M to the xy-plane, ψ being measured in the clock-wise sense. By the angles θ and ψ the position of g is perfectly determined. Assuming $\psi = 0$, when $\theta = 0$, and imposing upon ψ the condition $\psi = p/q \cdot \theta$, where p and q are relatively prime integers, the surface of the class, characterized by two definite values of p and q, may be represented parametrically by

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta, \qquad z = (\rho - a) \cot\frac{p}{q}\theta, \tag{1}$$

* "Rappresentazione di una classe di superficie gobbe sopra un piano, e determinazione delle loro curve assintotiche," *Annali di Matematica*, Ser. II, Vol. 1, pp. 248–258 (1867).

† "Intorno alla rappresentazione della superficie gobbe di genere $p = 0$," *Annali di Matematica*, Ser. II, Vol. 4, pp. 50–72 (1870).

‡ "On Certain Skew Surfaces, otherwise Scrolls," *Transactions of the Cambridge Philosophical Society*, Vol. XI, Part II, pp. 277–289 (1869).

§ "Ueber die geradlinigen Flächen fünften Grades," *Journal für reine und angewandte Mathematik*, Vol. 76, pp. 23–57 (1867).

|| "Ueber Flächen, welche Schaaren rationaler Curven besitzen," *Mathematische Annalen*, Vol. 3, pp. 161–227 (1871).

¶ "Ueber die geradlinigen Flächen vom Geschlechte $p = 0$," *Mathematische Annalen* Vol. 5, pp. 1–26 (1872).

** "Sur les surfaces algébriques dont toutes les sections planes sont unicursales," *Journal für reine und angewandte Mathematik*, Vol. 100, pp. 71–78 (1885).

in which a is the radius of C_2, ρ and θ are the parametric coördinates of a point of the surface. Since

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\tan^2\left(q\cdot\frac{\theta}{2q}\right)}{1+\tan^2\left(q\cdot\frac{\theta}{2q}\right)},$$

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2\cdot\tan\left(q\cdot\frac{\theta}{2q}\right)}{1+\tan^2\left(2q\frac{\theta}{q\cdot}\right)},$$

$$\cot\frac{p}{q}\theta = 1/\tan\left(2p\cdot\frac{\theta}{2q}\right),$$

may all be expressed as rational functions of the parameter $\tan\frac{\theta}{2q} = t$, x, y, z become rational functions of the parameters ρ and t, so that (1) *defines a rational ruled surface.*

To determine the order of the surface, the number of points of intersection of the line

$$a_1x = b_1y + c_1z + d_1 = 0, \qquad a_2x + b_2y + c_2z + d_2 = 0 \tag{2}$$

with the surface must be found. Substituting in (2) for x, y, z their expressions given by (1), and eliminating ρ, the equation

$$\{(a_1d_2 - a_2d_1)\cos\theta + (b_1d_2 - b_2d_1)\sin\theta\}\tan\frac{p}{q}\theta$$
$$-(a_1c_2 - a_2c_1)\cos\theta - (b_1c_2 - b_2c_1)\sin\theta + c_1d_2 - c_2d_1 = 0 \tag{3}$$

is obtained. Making use of the identities

$$1+\tan^2 m\theta = \cos^{2m}\theta(1+\tan^2\theta)^m/\cos^2 m\theta, \tag{4}$$

$$\sin m\alpha = \binom{m}{1}\cos^{m-1}\alpha\cdot\sin\alpha - \binom{m}{3}\cos^{m-3}\alpha\cdot\sin^3\alpha + \cdots, \tag{5}$$

$$\cos m\alpha = \binom{m}{0}\cos^{m}\alpha - \binom{m}{2}\cos^{m-2}\alpha\cdot\sin^2\alpha + \cdots, \tag{6}$$

and putting $\frac{\theta}{2q} = \alpha$, $\tan\alpha = t$,

$$\frac{\sin q\alpha}{\cos^q\alpha} = \binom{q}{1}\tan\alpha - \binom{q}{3}\tan^3\alpha + \cdots \pm \tan^{-q}\alpha = \phi^q(t),$$

$$\frac{\cos q\alpha}{\cos^q\alpha} = \binom{q}{0} - \binom{q}{2}\tan^2\alpha + \cdots \pm \binom{q}{q-1}\tan^{q-1}\alpha = \psi^{q-1}(t),$$

where ϕ and ψ are rational integral functions of t of order q and $q-1$, as indicated by the upper indices. In this manner we obtain for $\sin\theta$, $\cos\theta$, $\tan\frac{p}{q}\theta$ the expressions,

$$\sin\theta = \frac{2\phi^q(t)\psi^{q-1}(t)}{(1+t^2)^q}, \quad \cos\theta = \frac{\psi^{2q-2}(t)-\phi^{2q}(t)}{(1+t^2)^q}, \quad \tan\frac{p}{q}\theta = \frac{2\phi_1(t)\cdot\psi_1(t)}{\psi_1^2(t)-\phi_1^2(t)}. \tag{7}$$

The numerator of the expression for $\tan(p/q)\theta$ is of degree $2p-1$, the denominator of degree $2p$ in t. Substituting these expressions in (5), an equation of degree $2(p+q)$ in t is obtained, so that when q is odd, a line (2) cuts the surface in as many points. Hence, *when q is odd, the order of the surface is $2(p+q)$.*

When $q = 2s$ is even, so that in (5)

$$\tan\frac{p}{q}\theta = \tan\left(p\cdot\frac{\theta}{2s}\right),$$

then putting $\tan(\theta/2s) = t$, with p odd, there is, in analogy with (7),

$$\sin\theta = \frac{2\tan\left(s\cdot\frac{\theta}{2s}\right)}{1-\tan^2\left(s\cdot\frac{\theta}{2s}\right)} = \frac{2\phi^s(t)\cdot\psi^{s-1}(t)}{(1+t^2)^s},$$

$$\cos\theta = \frac{1-\tan^2\left(s\cdot\frac{\theta}{2s}\right)}{1+\tan^2\left(s\cdot\frac{\theta}{2s}\right)} = \frac{\psi^{2s-2}(t)-\phi^{2s}(t)}{\psi^{2s-2}(t)+\phi^{2s}(t)},$$

$$\tan\frac{p}{q}\theta = \tan\left(p\cdot\frac{\theta}{2s}\right) = \frac{\psi^p(t)}{\psi^{p-1}(t)},$$

in which the upper indices indicate again the degrees of the polynomials ϕ and ψ in t. Substituting these expressions in (3), an equation of degree $p+q$ in t is obtained, so that in this case the order of the surface is $p+q$. The results may be stated as

THEOREM 1. *The surface of the class is rational and of order $2(p+q)$ or $p+q$, according as q is odd or even.*

3. Cartesian and Homogeneous Equations of the Surface.

Applying the formula

$$\tan rw = \frac{\binom{r}{1}\tan w - \binom{r}{3}\tan^3 w + \binom{r}{5}\tan^5 w - \cdots}{\binom{r}{0} - \binom{r}{2}\tan^2 w + \binom{r}{4}\tan^4 w - \cdots}. \tag{8}$$

to the identity

$$\tan p\theta = \tan q \cdot \frac{p\theta}{q}, \tag{9}$$

we get

$$\frac{\binom{p}{1}\tan\theta - \binom{p}{3}\tan^3\theta + \binom{p}{5}\tan^5\theta - \cdots}{\binom{p}{0} - \binom{p}{2}\tan^2\theta + \binom{p}{4}\tan^4\theta - \binom{p}{6}\tan^6\theta + \cdots}$$

$$= \frac{\binom{q}{1}\tan\frac{p\theta}{q} - \binom{q}{3}\tan^3\frac{p\theta}{q} + \binom{q}{5}\tan^5\frac{p\theta}{q} - \cdots}{\binom{q}{0} - \binom{q}{2}\tan^2\frac{p\theta}{q} + \binom{q}{4}\tan^4\frac{p\theta}{q} - \binom{q}{6}\tan^6\frac{p\theta}{q} + \cdots}. \tag{10}$$

From (1)

$$\tan\theta = \frac{y}{x}, \qquad \tan\frac{p\theta}{q} = \frac{\rho - a}{z}.$$

Substituting this in (10), and assuming q odd, after some reduction, the cartesian equation of the surface is obtained in the form

$$\frac{\binom{p}{1}x^{p-1}y - \binom{p}{3}x^{p-3}y^3 + \binom{p}{5}x^{p-5}y^5 - \cdots}{\binom{p}{0}x^p - \binom{p}{2}x^{p-2}y^2 + \binom{p}{4}x^{p-4}y^4 - \cdots}$$

$$= \frac{\binom{q}{1}(\rho - a)z^{q-1} - \binom{q}{3}(\rho - a)^3 z^{q-3} + \cdots + (-1)^{(q+1)/2}(\rho - a)^{q-2}z^2 + (-1)^{(q-1)/2}(\rho - a)^q}{\binom{q}{0}z^q - \binom{q}{2}(\rho - a)^2 z^{q-2} + \cdots + (-1)^{(q+1)/2}(\rho - a)^{q-3}z^3 + (-1)^{(q-1)/2}(\rho - a)^{q-1}z}. \tag{11}$$

The expansions of $(\rho - a)^k$ bring in terms of the form $c_1(x^2 + y^2)^m$, $c_2(x^2 + y^2)^n\sqrt{x^2 + y^2}$, where c_1 and c_2 are definite constants. Hence, to rationalize the equation, all terms with $\sqrt{x^2 + y^2}$ as a factor must be collected on one side, all other terms on the other side of the equation. On squaring, a rational equation of order $2(p + q)$ is obtained, which is in agreement with the result of the foregoing section.

Monge* has shown that the equation of a ruled surface whose generatrices pass through the z-axis, has the form

$$z = x\psi\left(\frac{y}{x}\right) + \pi\left(\frac{y}{x}\right), \tag{12}$$

* "Application de l'Analyse à la Géométrie," 5 ed. (1850), pp. 83–89.

in which ψ and π are arbitrary functions (differentiable). This result he gained by differential geometry. Equation (11) can be readily put in the form (12). The left-hand member of (11) is a rational function of (y/x); the right-hand member a rational function of $(\rho - a)/z$. Consequently $(\rho - a)/z$ is an algebraic function of (y/x), say

$$(\rho - a)/z = R(y/x).$$

From this $\rho^2 = \{a + zR(y/x)\}^2$, or $x^2(1 + (y/x)^2) = \{a + zR(y/x)\}^2$. Extracting the square root, which we may assume positive, and solving for z, we have

$$z = x\frac{\sqrt{1 + (y/x)^2}}{R(y/x)} - \frac{a}{R(y/x)}, \tag{13}$$

so that Monge's proposition is verified for our class of surfaces.

Making use of the identities

$$\binom{r}{1}A^{r-1}B - \binom{r}{3}A^{r-3}B^3 + \binom{r}{5}A^{r-5}B^5 - \cdots$$
$$= -\frac{i}{2}\{(A + iB)^r - (A - iB)^r\} \tag{14}$$

$$\binom{r}{0}A^r - \binom{r}{2}A^{r-2}B^2 + \binom{r}{4}A^{r-4}B^4 - \cdots$$
$$= \frac{1}{2}\{(A + iB)^r + (A - iB)^r\} \tag{15}$$

and introducing the homogeneous isotropic coördinates $x_1 = x + iy$, $x_2 = x - iy$, $x_3 = as + iz$, $x_4 = as - iz$, where s denotes the variable making the system (x, y, z) homogeneous, equation (11) reduces to the symmetric irrational form

$$x_1x_2(x_1^{p/q} - x_2^{p/q})^2 - (x_3x_1^{p/q} - x_4x_2^{p/q})^2 = 0. \tag{16}$$

The intersection of this surface with the plane at infinity of the system x, y, z, s, is obtained by putting $s = 0$, so that $x_3 = iz, x_4 = -iz$. Applying the collineation $x_1/z = \xi$, $x_2/z = \eta$, the curve of intersection then assumes the form

$$\xi\eta(\xi^{p/q} - \eta^{p/q})^2 + (\xi^{p/q} + \eta^{p/q})^2 = 0.$$

where ξ and η are again cartesian coördinates. Introducing polar coördinates by putting $\xi = \rho e^{i\theta}$, $\eta = \rho e^{-i\theta}$, this reduces to the simple form

$$\rho = \tan\frac{p}{q}\theta. \tag{17}$$

From the cinematic definition of the surface it is apparent that this curve may be considered as the projection of the curve at infinity from the origin

upon the plane $z = 1$. That it is rational as proved before, is also apparent if we put $\theta/q = w$, then $\xi = \tan pw \cdot \cos qw$, $\eta = \tan pw \cdot \sin qw$ may be expressed rationally by the parameter $\tan(w/2)$.

To investigate the behavior of the curve at the isotropic points, we may write its equation in the rational projective form

$$\begin{aligned}(x_1^p + x_2^p)^2 x_1 x_2 \Bigg\{ \binom{q}{1} x_3^{q-1} - \binom{q}{3} x_1 x_2 x_3^{q-3} + \cdots \\ \pm \binom{q}{4} x_1^{(q-5)/2} x_2^{(q-5)/2} x_3^4 \mp \binom{q}{2} x_1^{(q-3)/2} x_2^{(q-3)/2} x_3^2 \pm x_1^{(q-1)/2} x_2^{(q-1)/2} \Bigg\}^2 \\ + (x_1^p - x_2^p)^2 x_3^2 \Bigg\{ \binom{q}{0} x_3^{q-1} - \binom{q}{2} x_1 x_2 x_3^{q-3} + \cdots \\ \mp \binom{q}{3} x_1^{(q-3)/2} x_2^{(q-3)/2} x_3^2 \pm \binom{q}{1} x_1^{(q-1)/2} x_1^{(q-1)/2} \Bigg\}^2,\end{aligned}$$

in which the algebraic signs within the brackets are alternating throughout.

"Placing the curve on the analytic triangle," the terms nearest the vertex $X_1(x_2 = 0,\ x_3 = 0)$, after a rather tedious calculation, are found in order

$$x_1^{2p+q} x_2^q + \binom{q}{1} x_1^{2p+q-1} x_2^{q-1} x_3^2 + \binom{q}{2} x_1^{2p+q-2} x_2^{q-2} x_3^4 + \cdots + x_1^{2p} x_3^{2q} + \cdots,$$

and lie on a straight line of the "analytic triangle." Putting $x_1 = 1$, this may be written as

$$(x_2 + x_3^2)^q + \cdots,$$

so that in the neighborhood of the isotropic point $(x - iy = 0,\ z = 0)$ the curve has the same singularity as the curve $(x_2 + x_3^2)^q = 0$, which represents a q-fold parabola. This point must therefore be counted as $q(q-1)$ double points. As the equation is symmetric with respect to x_1 and x_2, also the other isotropic point $(x_1 = 0,\ x_3 = 0)$, or $(x + iy = 0,\ z = 0)$ must be counted as $q(q-1)$ double-points. The isotropic points must therefore be counted as $2q(q-1)$ double points.

When $q = 2s$ is even, the equation of the surface, which may also be written in the form

$$x_1^p(x_4 + i\rho)^q - x_2^p(x_3 - i\rho)^q = 0,$$

or

$$\begin{aligned}x_1^p \left[x_4^q - \binom{q}{2} x_4^{q-2}\rho^2 + \binom{q}{4} x_4^{q-4}\rho^4 - \cdots \right] \\ - x_2^p \left[x_3^q - \binom{q}{2} x_3^{q-2}\rho^2 + \binom{q}{4} x_3^{q-4}\rho^4 - \cdots \right] \\ = -i \Bigg\{ x_1^p \left[\binom{q}{1} x_4^{q-1}\rho - \binom{q}{3} x_4^{q-3}\rho^3 + \cdots \right] \\ + x_2^p \left[\binom{q}{1} x_3^{q-1}\rho - \binom{q}{3} x_3^{q-3}\rho^3 + \cdots \right] \Bigg\},\end{aligned}$$

may be rationalized as follows:

Squaring both sides and making use of the identity

$$\left\{\binom{r}{0}A^r-\binom{r}{2}A^{r-2}B^2+\binom{r}{4}A^{r-4}B^4-\cdots\right\}^2$$
$$+\left\{\binom{r}{1}A^{r-1}B-\binom{r}{3}A^{r-3}B^3+\binom{r}{5}A^{r-5}B^5-\cdots\right\}^2=(A^2+B^2)^r,$$

we get

$$x_1^{2p}(x_4^2+\rho^2)^q+x_2^{2p}(x_3^2+\rho^2)^q$$
$$-2x_1^px_2^p\left\{\left[x_3^q-\binom{q}{2}x_3^{q-2}\rho^2+\cdots\right]\left[x_4^q-\binom{q}{2}x_4^{q-2}\rho^2+\cdots\right]\right.$$
$$\left.-\left[\binom{q}{1}x_3^{q-1}\rho-\binom{q}{3}x_3^{q-3}\rho^3+\cdots\right]\left[\binom{q}{1}x_4^{q-1}\rho-\binom{q}{3}x_4^{q-3}\rho^3+\cdots\right]\right\}=0. \quad (18)$$

The bracket-expression multiplying $2x_1^px_2^p$ may be written in the form

$$\tfrac{1}{4}[\{(x_3+i\rho)^q+(x_3-i\rho)^q\}\{(x_4+i\rho)^q+(x_4-i\rho)^q\}$$
$$+\{(x_3+i\rho)^q-(x_3-i\rho)^q\}\{(x_4+i\rho)^q-(x_4-i\rho)^q\}]$$
$$=\tfrac{1}{2}\{(x_3+i\rho)^q(x_4+i\rho)^q+(x_3-i\rho)^q(x_4-i\rho)^q\}.$$

Then, substracting $2x_1^px_2^p(x_3+\rho^2)^{q/2}(x_4^2+\rho^2)^{q/2}$ from the first two terms of (18), and adding the same expression to the third term, the equation may be written in the form

$$\{x_1^p(x_4^2+\rho^2)^{q/2}-x_2^p(x_3^2+\rho^2)^{q/2}\}^2$$
$$=x_1^px_2^p\{(x_3+i\rho)^{q/2}(x_4+i\rho)^{q/2}-(x_3-i\rho)^{q/2}(x_4-i\rho)^{q/2}\}^2. \quad (19)$$

Now, since $q=2s$, be expanding,

$$(x_3+i\rho)^s(x_4+i\rho)^s=x_3^sx_4^s+i\binom{s}{1}\rho\,(x_3^{s-1}x_4^s+x_3^sx_4^{s-1})$$
$$-i\rho^3\left[\binom{s}{1}\binom{s}{2}x_3^{s-1}x_4^{s-2}+\binom{s}{1}\binom{s}{2}x_3^{s-2}x_4^{s-1}+\binom{s}{3}x_3^sx_4^{s-3}+\binom{s}{3}x_3^{s-3}x_4^s\right]$$
$$+\cdots.$$

The expansion of $(x_3-i\rho)^s(x_4-i\rho)^s$ is obtained from the foregoing by replacing i by $-i$. so that in the bracket-expression, multiplying $x_1^px_2^p$ in (19), all terms without the factorf$\pm\, i$ cancel, leaving for it

$$2i\left\{\binom{s}{1}\rho(x_3^{s-1}x_4^s+x_3^sx_4^{s-1})-\rho^3\left[\binom{s}{1}\binom{s}{2}(x_3^{s-1}x_4^{s-2}+x_3^{s-2}x_4^{s-1}\right.\right.$$
$$\left.\left.+\binom{s}{3}(x_3^sx_4^{s-3}+x_3^{s-3}x_4^s)\right]+\cdots\right\}.$$

Taking out the factor ρ, and noting that $\rho^2 = x_1x_2$, (19) now assumes the form

$$\{x_1^p(x_4^2 + x_1x_2)^s - x_2^p(x_3^2 + x_1x_2)^s\}^2 = -4(x_1x_2)^{p+1}\left\{\binom{s}{1}(x_3^{s-1}x_4^s + x_3^s x_4^{s-1}) - x_1x_2\left[\binom{s}{1}\binom{s}{2}(x_3^{s-1}x_4^{s-2} + x_3^{s-2}x_4^{s-1}) + \binom{s}{3}(x_3^s x_4^{s-3} + x_3^{s-3}x_4^3)\right] + \cdots\right\}^2.$$

Extracting the square root on both sides of the equation which, according to geometric tests in examples given hereafter must be taken with the positive sign, we finally get for the equation of the surface, when q is even,

$$x_1^p(x_4^2 + x_1x_2)^s - x_2^p(x_3^2 + x_1x_2)^s - 2i(x_1x_2)^{p+1/2}\left\{\binom{s}{1}(x_3^{s-1}x_4^s + x_3^s x_4^{s-1}) - x_1x_2\left[\binom{s}{1}\binom{s}{2}(x_3^{s-1}x_4^{s-2} + x_3^{s-2}x_4^{s-1}) + \binom{s}{3}(x_3^s x_4^{s-3} + x_3^{s-3}x_4^s)\right] + \cdots\right\} = 0. \quad (20)$$

As p is odd, $p + 1$ is even, and $(p + 1)/2$ an integer, so that equation (20) is of degree $p + 2s = p + q$, and the surface consequently of order $p + q$, as was proved before in the parametric form. From this the cartesian form is easily obtained by substituting for x_1, x_2, x_3, x_4 their expressions in terms of x, y, z.

As in the case of q odd, the intersection of the surface with the plane at infinity can be placed upon the analytic triangle. By the same method is found that the isotropic points must be counted as $2s(s - 1)$ double points of the infinite curve of intersection in case of q even.

4. (α, β)-CORRESPONDENCE BETWEEN C_1 AND C_2 DETERMINED BY THE GENERATRICES OF THE SURFACE.

When we put $z = 0$ in (11), the intersection of the surface with the xy-plane is obtained

$$\left\{\binom{p}{0}x^p - \binom{p}{2}x^{p-2}y^2 + \binom{p}{4}x^{p-4}y^4 - \cdots\right\}(\rho - a)^q = 0. \quad (21)$$

To make this equation rational, put the bracket-expression equal to V, then $V^{1/q}(\rho - a) = 0$, $V^{2/q}(x^2 + y^2) = a^2V^{2/q}$, and from this as the equation of the intersection

$$V^2(x^2 + y^2 - a^2)^q = 0, \quad (22)$$

which shows that the directrix-circle C_2 is a q-fold curve of the surface.

when q is odd. The rest of the intersection consists of the p double generatrices in the xy-plane defined by $V^2 = 0$. These double lines divide the full angle into $2p$ equal parts, as is seen by actual determination of the angles from $V \equiv \rho^p(e^{ip\theta} + e^{-ip\theta}) = 0$, or $e^{2ip\theta} = -1$. From this is found $\theta = (k\pi/p) - (\pi/4p)$, so that θ has p values incongruent to π, determined by $k = 1, 2, 3, \cdots, p$. When q is odd, the midpoint M of the generatrix g describes C_2 q times, so that θ increases by $q \cdot 2\pi$ and $(p/q)\theta$ by $p \cdot 2\pi$. From this follows that g turns around p times in the plane e and sweeps $2p$ times through the z-axis, so that the z-axis or C_1 is a $2p$ fold line of the surface. To determine the positions of the generatrices in a plane e determined by an angle θ more definitely, let g_0 be the initial position of the generatrix determined by the angles θ and $\psi = (p/q)\theta$. As θ increases by π, e turns through an angle π and the generatrix moves to the position g_1 making an angle $(p/q)\theta + (p/q)\pi$ with the perpendicular to the xy-plane through the midpoint M_1 of g_1. As $\theta + \pi$ increases again by π, the generatrix g_1 moves to the new position g_2 through M, determined by the angle $(p/q)\theta + (p/q)2\pi$. While θ increases from 0 to $q \cdot 2\pi$, the generatrix describes the entire surface and returns point for point to its initial position g_0. Denoting the $2q$ positions of the generatrix by $g_0, g_1, g_2, \cdots, g_{2q-1}$ the corresponding determining angles may be listed in the table:

Through M		Through M_1	
g_0	$p/q\theta$	g_1	$p/q(\theta + \pi)$
g_2	$p/q(\theta + 2\pi)$	g_3	$p/q(\theta + 3\pi)$
g_4	$p/q(\theta + 4\pi)$	g_5	$p/q(\theta + 5\pi)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
g_{2q-2}	$p/q\{\theta + (2q - 2)\pi\}$	g_{2q-3}	$p/q\{\theta + (2q - 3)\pi\}$
g_{2q}	$p/q\{\theta + 2q\pi\}$	g_{2q-1}	$p/q\{\theta + (2q - 1)\pi\}$

Through each M and M_1 there are q generatrices which divide the full angle around M as well as around M_1 in the plane e into $2q$ equal parts. Through every point of C_2 there are q generatrices.

To determine the number of generatrices through a point of C_1, let such a point be determined by $z =$ constant, then, $a\rho = 0$, from (1)

$$\cot \frac{p}{q}\theta = -\frac{z}{a} = \text{constant}. \tag{23}$$

From this $\frac{p}{q}\theta - k\pi = -\operatorname{arccot}\frac{z}{a}$, and

$$\theta = \frac{kq}{p}\pi - \frac{q}{p}\operatorname{arccot}\frac{z}{a}, \qquad \left(\operatorname{arccot}\frac{z}{a} \leqq \frac{\pi}{2}\right) \tag{24}$$

for $\theta \leqq \theta \leqq q \cdot 2\pi$. This gives for θ $2p$ values, determined by $k = 1, 2, 3, \cdots, 2p$, and incongruent $q \cdot 2\pi$, and consequently $2p$ generatrices through a point of the z-axis, or C_1. These lie in pairs of symmetric lines in planes through C_1, each pair being determined by two values k and k_1, such that their difference $k_1 - k = p$.

From the fact, that to every point of C_1 correspond $2p$ points of C_2, the points of intersection of the $2p$ generatrices through the point on C_1; and to every point of C_2 q points of C_1, the intersections of the q generatrices through the point on the C_2 with C_1, it is apparent that the generatrices cut out on C_1 and C_2 an (α, β)-correspondence of the specific form $(q, 2p)$.

To establish the algebraic form of this correspondence, we may represent C_2 parametrically by

$$x = a\frac{1-\lambda^2}{1+\lambda^2}, \qquad y = a\frac{2\lambda}{1+\lambda^2},$$

so that conversely $\lambda = y/(a + x)$. If we now put $\tan(\theta/2q) = t$, then for $\rho = 0$,

$$z = -a\cot\frac{p}{q}\theta = -a\cot 2p\cdot\frac{\theta}{2q} = F(t), \tag{25}$$

$$\lambda = \frac{\sin\theta}{1+\cos\theta} = \frac{2\tan q\cdot\dfrac{\theta}{2q}}{1-\tan^2 q\cdot\dfrac{\theta}{2q}} = G(t), \tag{26}$$

both rational functions of t. In $F(t)$ the highest power of t is $2p$, in $G(t)$ it is q. The elimination of t between (25) and (26) leads to a polynomial relation between z and λ which is of degree q in z, and of degree $2p$ in λ, and of degree $2p + q$ in both, and which is of deficiency 0. It has the form

$$\lambda^{2p}(a_0z^q + a_1z^{q-1} + \cdots) + \lambda^{2p-1}(b_0z^q + b_1z^{q-1} + \cdots) + \cdots + w_0z^q + w_1z^{q-1} + \cdots = 0. \tag{27}$$

This, when considered as a curve in a (λ, z)-plane, being of deficiency 0, has $\frac{1}{2}(2p + q - 2)(2p + q - 1)$ double-points, or double roots. But the infinite points of the λ and z-axis, are in the same order q and $2p$-fold points so that the number of finite double points is

$$\tfrac{1}{2}\{(2p + q - 2)(2p + q - 1) - 2p(2p - 1) - q(q - 1)\} = 2pq - 2p - q + 1. \tag{28}$$

But there are no real double generatrices which are simultaneously double lines of the system of $2p$ generatrices through a point of the z-axis and the q generatrices through a point of C_2. The surface has therefore a number of imaginary double-generatrices as given by (28). Hence

THEOREM 2. *When q is odd, the generatrices of the surface cut C_1 and C_2 in two point sets which are in a $(q, 2p)$-correspondence. C_1 and C_2 are $2p$-fold and q-fold curves of the surface. The surface has moreover p real and $2pq - 2p - q + 1$ imaginary double generatrices.*

When $q = 2s$ is even, $\psi = (p/2s)\theta$ increases from 0 to $p\pi$, when θ increases from 0 to $s \cdot 2\pi$. The midpoint M of the generatrix g turns s times about the z-axis, and g turns p times around M in the plane e. From this follows that C_1 and C_2 are respectively p- and s-fold lines of the surface. When θ increases from θ_0 to $\theta_0 + s \cdot 2\pi$, the generatrix g moves from the initial position g_0 to the position g_{2s} which coincides with g_0, but the segments into which M divides g_0 are interchanged on g_{2s}. In other words g describes a closed unifacial ruled surface.

From a point of C_1 there are p generatrices cutting C_2. This appears from the formula

$$\theta = \frac{kq}{p}\pi - \frac{q}{p}\operatorname{arccot}\frac{z}{a} \qquad \left(\operatorname{arccot}\frac{z}{a} \leqq \frac{\pi}{2}\right),$$

with $0 \leqq \theta \leqq s \cdot 2\pi$, which determines p values for θ by putting $k = 1, 2, 3, \cdots, p$. For two distinct values k and k_1 of this set we have $\theta_1 - \theta = [(k_1 - k)/p]q\pi$. As p is odd, q even, and $k_1 - k < p$, and p and q are relatively prime, $\theta_1 - \theta$ can never be an odd multiple of π. Hence no two of the generatrices through a point of C_1 can lie in the same plane e; hence the surface has no real double generatrices.

Through every point of C_1 there are p generatrices cutting C_2 in as many points; through every point of C_2 there are s generatrices cutting C_1 in the same number of points. Hence

THEOREM 3. *When $q = 2s$ is even the, generatrices of the surface cut C_1 and C_2 in two point sets which are in a rational (s, p)-correspondence. C_1 and C_2 are respectively p- and s-fold curves of the surface. The surface has no real, but has $ps - p - s + 1$ imaginary double generatrices.*

5. DOUBLE CURVES OF THE SURFACE WHEN q IS ODD.

The $2q$ generatrices in a plane e intersect in q^2 points D_{ik} of the double curve of the surface. Each pair of indices ik consists throughout of an odd and an even number. These q^2 couples may be arranged into $(q + 1)/2$ cyclic groups of which $(q - 1)/2$ are of order $2q$, and 1 is of order q. Those of order $2q$ are

$$\begin{array}{ccccc}
01 & 03 & 05 & \cdots & 0 \cdot q - 2 \\
12 & 14 & 16 & \cdots & 1 \cdot q - 1 \\
23 & 25 & 27 & \cdots & 2 \cdot q. \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
2q - 1 \cdot 2q & 2q - 1 \cdot 2 & 2q - 1 \cdot 4 & \cdots & 2q - 1 \cdot q - 3,
\end{array}$$

that of order q is

$$\begin{array}{c} 0q \\ 1\cdot q+1 \\ 2\cdot q+2 \\ \vdots \\ q-1\cdot 2q-1. \end{array}$$

In this group the generatrices g_0 and g_q, intersecting at D_{0q}, are determined by the angles $(p/q)\theta$ and $(p/q)(\theta + q\pi) = (p/q)\theta + p\pi$, which shows that when θ varies, the corresponding generatrices g_0 and g_q always intersect in a point of C_1, i.e., the intersections D of the group of order q all lie on C_1. On the other hand the intersections of each group of order $2q$ are cyclically permuted when θ increases by $q\cdot 2\pi$, so that the points D_{01}, D_{03}, D_{05}, $\cdots$ $D_{0\cdot q-1}$ describe $(q-1)/2$ twisted curves which together form the double curve of the ruled surface.

To find the equation of any of these double curves, for example the one described by the point starting from $D_{0\cdot 2k-1}$, where k may have any value between 1 and $(q-1)/2$, let x, y, z be the cartesian coördinates of $D_{0\cdot 2k-1}$, and ρ, θ the polar coördinates of x, y. The equations of g_0 and $g_{0\cdot 2k-1}$ in their planes (ρ, z) are

$$z = (\rho - a)\cot\frac{p}{q}\theta, \qquad z = -(\rho - a)\cot\frac{p}{q}\{\theta + (2k-1)\pi\}.$$

From this

$$\rho = \frac{a\sin\frac{p}{q}(2k-1)\pi}{\sin\frac{p}{q}\{2\theta + (2k-1)\pi\}}; \tag{29}$$

which is the polar equation of the projection of the double curve upon the xy-plane. As $x = \rho\cos\theta$, $y = \rho\sin\theta$, the parametric (θ) equations of the double curve are

$$\begin{aligned} x &= \frac{a\sin\frac{p}{q}(2k-1)\pi\cdot\cos\theta}{\sin\frac{p}{q}\{2\theta + (2k-1)\pi\}}, \\ y &= \frac{a\sin\frac{p}{q}(2k-1)\pi\cdot\sin\theta}{\sin\frac{p}{q}\{2\theta + (2k-1)\pi\}}, \\ z &= -a\frac{\cos\frac{p}{q}\{2\theta + (2k-1)\pi\} + \cos\frac{p}{q}(2k-1)\pi}{\sin\frac{p}{q}\{2\theta + (2k-1)\pi\}}. \end{aligned} \tag{30}$$

From this all double curves are obtained by putting successively $k = 1, 2, 3, \cdots, (q-1)/2$.

Eliminating θ between (29) and the last equation of (30), the relation between ρ and z becomes

$$(\rho^2 + z^2 - a^2)\sin\frac{p}{q}(2k-1)\pi = 2\rho z\cos\frac{p}{q}(2k-1)\pi, \tag{31}$$

which shows that the $2q$ intersections of the generatrices of a group in any plane e through the z-axis lie on two hyperbolas, which are symmetrical with respect to the z-axis, since to every increase of θ by π corresponds a rotation of the hyperbola (31) about the z-axis through an angle π. Thus the points $D_{01}, D_{23}, D_{45}, \cdots$ lie on (14), while $D_{12}, D_{34}, D_{56}, \cdots$ lie on the reflexion of (14). The double curve belonging to the group lies therefore on a surface of revolution of order 4 with the equation

$$(x^2+y^2-z^2-a^2)^2\sin^2\frac{p}{q}(2k-1)\pi - 4(x^2+y^2)z^2\cos^2\frac{p}{q}(2k-1)\pi = 0, \tag{32}$$

with C_2 as a double curve.

The asymptotes of the hyperbola (31) are determined by the equation of their slope z/ρ,

$$\left(\frac{z}{\rho}\right)^2 + 2\left(\frac{z}{\rho}\right)\cot\frac{p}{q}(2k-1)\pi - 1 = 0,$$

which shows that the product of the slopes of the asymptotes is -1. (31) is therefore an equilateral hyperbola.

To find the generatrices of the surface with the same slopes, in other words the infinite points of the double curve described by $D_{0\cdot 2k-1}$, the conditions must be satisfied:

(a) $$\cot\frac{p}{q}\theta = \tan\frac{p(2k-1)\pi}{2q},$$

and from this

$$\theta = \frac{q(2l+1)}{2p}\pi - \frac{(2k-1)\pi}{2},$$

for all values $l = 0, 1, 2, \cdots, 2p-1$. Two consecutive values of θ differ by the amount $(q/2p)\pi$. For every value of θ of this set there is a value $\theta' = \theta + q\pi$. For a definite value of l the angles are

$$\theta' = \frac{q\{2(l+p)+1\}}{2p}\pi - \frac{2k-1}{2}\pi \text{ and } \theta = \frac{q(2l+1)}{2p}\pi - \frac{2k-1}{2}\pi.$$

Hence there are $2p$ values of θ due to condition (1) for which the double curve has infinite points. The condition

(b) $$\cot\frac{p}{q}\theta = -\cot\frac{p}{2q}(2k-1)\pi$$

is satisfied, when

$$\theta = \frac{ql\pi}{p} - \frac{2k-1}{2}\pi,$$

which again determines $2p$ values for θ, for $l = 1, 2, 3, \cdots, 2p-1$. The values of θ determined by (a) and (b) are also obtained from the condition that in (29) $\rho = \infty$. This is the case for $(p/q)\{2\theta + (2k-1)\pi\} = m\pi$, or

$$\theta = \frac{mq\pi}{2p} - \frac{(2k-1)\pi}{2},$$

for $m = 1, 2, 3, \cdots, 4p$, and $k \leqq (q-1)/2$. For $k = (q+1)/2$ and $m = 1$, we get $\theta = 0$ and $\rho = a$. This corresponds to the one group of order q.

The order of the double curve may be determined by expressing sines and cosines in (30) by tangents. In the first place let $\theta + [(2k-1)/2]\pi = \phi$, or $\theta = -\{(2k-1)(\pi/2) - \phi\}$, so that $\sin\theta = (-1)^k \cos\phi$, $\cos\theta = (-1)^{k+1}$ $\sin\phi$. Then (30) may be written in the form

$$\begin{aligned} x &= a \cdot \frac{(-1)^{k+1} \sin\frac{p}{q}(2k-1)\pi \cdot \sin\phi}{\sin\frac{p}{q}2\phi}, \\ y &= a \cdot \frac{(-1)^{k} \sin\frac{p}{q}(2k-1)\pi \cos\phi}{\sin\frac{p}{q}2\phi}, \\ z &= -a \cdot \frac{\cos\frac{p}{q}2\phi + \cos\frac{p}{q}(2k-1)\pi}{\sin\frac{p}{q}2\phi}. \end{aligned} \tag{33}$$

Expressing sines and cosines in terms of $\tan(\theta/2q) = t$ by means of identities as given in (9), and using similar notations by functions ϕ, ψ, ϕ_1, ψ_1, of t the parametric equations of the double curve become

$$\begin{aligned} x &= (-1)^{k+1} a \sin\frac{p}{q}(2k-1)\pi \frac{\phi^q(t)\psi^{q-1}(t)(1+t^2)^{2p-q}}{\phi^{2p}(t)\cdot\psi^{2p-1}(t)}, \\ y &= (-1)^{k} a \sin\frac{p}{q}(2k-1)\pi \frac{\{\psi^{2q-1}(t) - \phi^{2q}(t)\}(1+t^2)^{2p-q}}{2\phi^{2p}(t)\cdot\psi^{2p-1}(t)}, \\ z &= -a\, \frac{\psi^{4p-2}(t) - \phi^{4p}(t) + \cos\frac{q}{p}(2k-1)\pi\cdot(1+t^2)^{2p}}{2\phi^{2p}(t)\cdot\psi^{2p-1}(t)}, \end{aligned} \tag{34}$$

which are rational functions of t. The highest power of t in these functions is $4p$, or $2q$, according as $q \lesseqgtr 2p$. Hence

THEOREM 4. *The order of each of the $(q-1)/2$ double curves is $4p$ or $2q$ according as $q \lesseqgtr 2p$, and when q is odd. They are rational and each lies on a surface of revolution of order 4 generated by the rotation of an equilateral hyperbola about the z-axis.*

As the surfaces of the class are rational, all plain sections are rational and must therefore have the maximum number of double points. We shall verify this number in case of q odd, and $q \lesseqgtr 2p$, for the curve of intersection with the plane at infinity. Consider first a plane $z =$ const., cutting the z-axis in a point Z. A pair of generatrices g and g' in the same plane e and through the same point U (different from Z) of the z-axis cut the plane $z =$ const. in two points A and A', which are collinear with Z. Now as θ increases, A and A' describe curves in central symmetry with Z, which will touch each other at Z, i.e., the form or tac-nod at Z. But Z must be counted as a $2p$-fold point of the curve of intersection with the plane $z =$ const., since the z-axis is a $2p$-fold line of the surface, and as it occurs p times that two branches of the curve form tac-nods, the point Z must be counted as

$$\frac{2p(2p-1)}{2} + p = 2p^2$$

double points. As this is true for every point of the z-axis, the latter must be counted as $2p^2$ double lines. The surface has therefor $2p^2 + p$ real double lines (p double lines in the xy-plane. The double points of the infinite curve of intersection are therefore made up of those absorbed by the isotropic points, of the intersections with the real double lines, with the imaginary double-lines, and with the $(q-1)/2$ double curves, each with $4p$ infinite points. These numbers are in the same order, and summed up:

$$\{2q(q-1)\} + \{2p^2 + p\} + \{2pq - 2p - q + 1\} + \left\{\frac{q-1}{2} \cdot 4p\right\} = \frac{\{2(p+q)-1\}\{2(p+q)-2\}}{2},$$

as required of a plain curve of order $2(p+q)$.

6. DOUBLE CURVES OF THE SURFACE WHEN g IS EVEN.

When $q = 2s$, there are two pencils, of s generatrices each, in every plane section through the z-axis, which intersect in s^2 points. When the plane e turns s times about the z-axis, starting from an initial position e_0 containing the initial position g_0 of the generatrix g, assuming in succession positions determined by the angles $\theta = 0, \pi, 2\pi, 3\pi, \cdots$, g will occupy the positions $g_0, g_1, g_2, \cdots, g_{2s-1}$. In this movement the intersection of the generatrices of even and odd indices in the same plane describe double curves of the surface. Every double curve, with the exception of one in

case when s is odd, cuts the initial plane e_0 in $2s$ points. When s is odd the number of intersections of the one curve with the plane is s, and as

$$s^2 = \frac{s-1}{2} \cdot 2s + s,$$

there are in this case $(s-1)/2$ double curves with $2s$ intersections, and one curve with s intersections with the plane e_0; i.e., altogether $(s+1)/2$ double curves. To each of these curves corresponds a cyclic group of points of intersection with the plane e_0. Denoting the groups of order $2s$ by $D_1, D_2, D_3, \cdots, D_{(s-1)/2}$ and the group of order s by $D_{(s-1)/2}$ the following table of these groups may be set up:

D_1	D_2	D_3	$\cdots$	$D_{(s-1)/2}$	$D_{(+1s)/2}$
01	03	05	$\cdots$	$0\cdot(s-2)$	$0\cdot(s)$
12	14	16	$\cdots$	$1\cdot(s-1)$	$1\cdot(s+1)$
23	25	27	$\cdots$	$2\cdot(s+1)$	$2\cdot(s+2)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$(2s-2)\cdot(2s-1)$	$(2s-2)\cdot 1$	$(2s-2)\cdot 3$	$\cdots$	$(2s-2)\cdot(s-4)$	$(s-1)\cdot[2(s-1)+1]$
$(2s-1)\cdot 0$	$(2s-1)\cdot 2$	$(2s-1)\cdot 4$	$\cdots$	$(2s-1)\cdot(s-3)$	

When $s = 2\sigma$ is even, then

$$s^2 = 4\sigma^2 = \sigma\cdot 4\sigma = \sigma\cdot 2s.$$

There are σ double curves, each with $2s$ intersections with the initial plane. The table of groups is in this case:

D_1	D_2	D_3	$\cdots$	D_σ
01	03	05	$\cdots$	$0\cdot(s-1)$
12	14	16	$\cdots$	$1\cdot(s)$
23	25	27	$\cdots$	$2\cdot(s+1)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$(2s-2)\cdot(2s-1)$	$(2s-2)\cdot 1$	$(2s-2)\cdot 3$	$\cdots$	$(2s-2)\cdot(s-3)$
$(2s-1)\cdot 0$	$(2s-1)\cdot 2$	$(2s-1)\cdot 4$	$\cdots$	$(2s-1)\cdot(s-2)$

The parametric equations of the double curves are obtained from those of (33) by putting $\tan(\phi/2s) = t$, so that in analogy with previous results

$$\sin\phi = \frac{2\tan s\cdot\dfrac{\phi}{2s}}{1+\tan^2 s\cdot\dfrac{\phi}{2s}} = \frac{2\phi^s(t)\cdot\psi^{s-1}(t)}{(1+t^2)^s},$$

$$\cos\phi = \frac{1-\tan^2 s\cdot\dfrac{\phi}{2s}}{1+\tan^2 s\cdot\dfrac{\phi}{2s}} = \frac{\psi^{2(s-1)}(t)-\phi^{2s}(t)}{(1+t^2)^s},$$

$$\cos\frac{p}{s}\phi = \frac{1-\tan^2 p\cdot\dfrac{\phi}{2s}}{1+\tan^2 p\cdot\dfrac{\phi}{2s}} = \frac{\phi_1^{2(p-1)}(t)-\psi_1^{2p}(t)}{(1+t^2)^p},$$

and the parametric equations of the double curve are

$$\begin{aligned} x &= (-1)^{k+1}a\sin\frac{p}{2s}(2k-1)\pi\,\frac{\phi^s(t)\psi^{s-1}(t)(1+t^2)^{p-s}}{\phi_1^p(t)\cdot\psi_1^{p-1}(t)},\\ y &= (-1)^k a\sin\frac{p}{2s}(2k-1)\pi\,\frac{\{\psi^{2(s-1)}(t)-\phi^{2s}(t)\}(1+t^2)^{p-s}}{2\phi_1^p(t)\cdot\psi_1^{p-1}(t)}, \qquad (35)\\ z &= -a\,\frac{\phi_1^{2(p-1)}(t)-\psi_1^{2p}(t)+\cos\frac{p}{2s}(2k-1)\pi\cdot(1+t^2)^{p-s}}{2\phi_1^p(t)\cdot\psi_1^{p-1}(t)}.\end{aligned}$$

When $p > s$, or $q < 2p$, and $2k-1 \neq s$, then the highest power of t in these expressions is t^{2p}, so that in this case the double curves are of order $2p$. When $p < s$, then they are of order $2s = q$.

In case of the double curve of order s, which is obtained from (33) by putting $2k-1 = s$, when s is odd, the coördinates may be expressed parametrically by $t = \tan(\phi/s)$ as rational functions of t, in which the highest power of t is t^p or t^s, according as $p > s$, or $p < s$, so that the double curve $[D(s+1)/2]$ is accordingly of order p or s, as appears from the parametric equations of the surface

$$\begin{aligned} x &= \pm\frac{(1+t^2)^{(p-s)/2}\left[\binom{s}{1}-\binom{s}{3}(1+t^2)t^2+\cdots\pm t^{s-1}\right]}{\binom{p}{1}-\binom{p}{3}(1+t^2)t^2+\cdots\pm t^{p-1}},\\ y &= \pm\frac{(1+t^2)^{(p-s)/2}\left[1-\binom{s}{2}t^2+\cdots\pm t^{s-1}\right]}{\binom{p}{1}t-\binom{p}{3}t^3+\cdots\pm t^p}, \qquad (36)\\ z &= -a\cot\left(\frac{p}{s}\phi\right) = f(t),\end{aligned}$$

where the signs within the brackets and in the denominators are alternating throughout, and $f(t)$ denotes a rational function in which the highest power of t is t^p. Hence

THEOREM 5. *When $q = 2s$ is even and s odd, there are $(s - 1)/2$ double curves of order $2p$ or q according as $p > s$ or $p < s$, and one double curve of order p or s, according as $p > s$ or $p < s$. When $s = 2\sigma$ is even, there are σ double curves of order $2p$ or q, according as $p > s$ or $p < s$.*

7. APPLICABILITY AMONG THE RULED SURFACES OF THE CLASS.*

Consider first the case of an odd q. Let the generatrix g whose midpoint M moves along the directrix circle C_2 generate the surface determined by the numbers p and q. At every position of M draw in the xy-plane an external tangent circle C_2' to C_2 with the radius $(m/n)a$. Through the center $0'$ of C_2' draw a perpendicular z' to the plane of C_2', and let g in every position be asscoiated with C_2' and z', just as it is associated with C_2 and z. Thus, the generatrices g associated with C_2', z' will generate a ruled surface F' whose directrix circle is C_2' and whose directrix line is z'. To determine the nature of this surface, we must determine how many revolutions C_2' has to make about the axes z' and z respectively, before the initial point M_0 on C_2', after a certain number of revolutions about the z'-axis, and the initial generatrix g_0 associated with F', after rotating a certain number of times about M in the plane e', return to the initial positions of M on C_2, and g_0 associated with the given surface. This will be the case when a certain multiple ν of the circumference of C_2' is equal to a certain multiple of q times the circumference of C_2, i.e., $\nu \cdot 2(m/n)a\pi = \mu q \cdot 2a\pi$. This gives for μ/ν the ratio

$$\frac{\mu}{\nu} = \frac{m}{nq}.$$

Assuming m and n, also m and q as relative primes, we may put $\mu = m$, $\nu = nq$. From this is seen that the ruled surface F' is generated by a generatrix whose midpoint describes C_2' nq times. The surface F' turns mq times about the z-axis. Moreover g associated with F' makes mp complete revolutions in the plane e' through the z'-axis. The order of the surface is therefore $2(mp + nq)$. Hence

THEOREM 6. *Surfaces of the class are applicable to each other when their orders are $2(p + q)$ and $2(mp + nq)$, and their radii of C_2 respectively a and $(m/n)a$, when q is odd, p and q, m and n, and m and q are relative primes.* This is still true when either $p = q = 1$, or $m = n = 1$, hence

Corollary 7. *A surface of the class of order $2(p + q)$, q odd, is applicable upon a surface of the class of order 4.*

Corollary 8. *A surface of the class of order $2(p + q)$, q odd, is applicable to another surface of the same class and order.*

* See Eisenhart, loc. cit., pp. 342–347.

When q is even, $q = 2s$, then we have under similar conditions

$$\frac{\mu}{\nu} = \frac{m}{ns},$$

and $mp + nq$ as the order of the surface. As m and p are odd, and q is even, $mp + nq$ is an odd number, so that we may state

THEOREM 9. *Surfaces of the class of odd order are applicable to each other when their orders are $p + q$ and $mp + nq$, and their radii a and $(m/n)a$, respectively, and when q is even and p and m are odd.*

As the surfaces of even and odd order are bifacial and unifacial respectively, we have

Corollary 10. *Bifacial and unifacial surfaces of the class are applicable to surfaces of the same type only.*

Similar results would be obtained by choosing for C_2' an internal tangent circle of C_2.

8. INTERSECTIONS OF THE SURFACES OF THE CLASS WITH A TORUS.

A torus whose circular axis coincides with the directrix circle C_2 of the surface cuts the surface in a composite curve of order $8(p + q)$ or $4(p + q)$, according as q is odd or even. If the radius of a meridian cross-section of the torus is b, then it follows easily that a part of this composite curve, generated by a point P on the generatrix g whose distance $PM = b$ from M is constant may be represented parametrically by

$$\begin{aligned} x &= \left(a + b \sin\frac{p}{q}\theta\right)\cos\theta, \\ y &= \left(a + b \sin\frac{p}{q}\theta\right)\sin\theta, \\ z &= b\cos\frac{p}{q}\theta. \end{aligned} \tag{37}$$

Putting again $\tan(\theta/2q) = t$, x, y, z may be expressed as rational functions of t, in which the highest power of t is $t^{2(p+q)}$, so that the order of the curve is $2(p + q)$. The polar equation of the projection of the curve upon the xy-plane is

$$\rho = a + b\sin\frac{p}{q}\theta, \tag{38}$$

which represents a socalled cyclo-harmonic curve,* which, in general, is of order $2(p + q)$, and is rational.

* R. E. Moritz, "On the Construction of Certain Curves Given in Polar Coördinates, *The American Mathematical Monthly*, Vol. XXIV, pp. 213–220 (1917).

When $a = 0$, and one of the integers p, q is even, the other odd, the curve is still of order $2(p + q)$; but when both are odd, then, as is well known,* the curve is of order $p + q$.

Expanding the identity $\sin p\theta = \sin q\cdot(p\theta/q)$, and substituting

$$\sin\frac{p}{q} = \frac{\rho - a}{b}, \qquad \sin\theta = \frac{y}{\rho}, \qquad \cos\theta = \frac{x}{\rho}, \qquad \cos\frac{p}{q}\theta = \frac{\sqrt{b^2 - (\rho - a)^2}}{b},$$

the cartesian equation of the curve (38) may be written in the form

$$\begin{aligned} b^q\left\{\binom{p}{1}x^{p-1}y - \binom{p}{3}x^{p-3}y^3 + \binom{p}{5}x^{p-5}y^5 - \cdots\right\} \\ = \rho^p\left\{\binom{q}{1}[b^2 - (\rho - a)^2]^{(q-1)/2}(\rho - a)\right. \\ \left. - \binom{q}{3}[b^2 - (\rho - a)^2]^{(q-3)/2}(\rho - a)^3 + \cdots\right\}. \end{aligned} \tag{39}$$

9. EXAMPLES.

A. Bifacial Surfaces.

1. *Quartic,*† $p = 1$, $q = 1$.

$$(x^2 + y^2)x^2 - (ax + yz)^2 = 0.$$

C_2 is a single curve, C_1 must be counted as two double lines. The y-axis is the only double line in the xy-plane.

2. *Sextic,* $p = 2$, $q = 1$.

$$(x^2 + y^2)(x^2 - y^2)^2 - \{2xyz - a(x^2 - y^2)\}^2 = 0.$$

C_2 is a single curve, C_1 must be counted as eight double lines, and there are two double lines in the xy-plane with the equations $x \pm y = 0$.

B. Unifacial Surfaces.

3. *Cubic,* $p = 1$, $q = 2$

$$\{(x - a)^2 - (z - y)^2\}y + 2x(x - a)(z - y) = 0,$$

or in parametric form

$$x = \rho\frac{1 - t^2}{1 + t^2}, \qquad y = \rho\frac{2t}{1 + t^2}, \qquad z = \frac{\rho - a}{t}.$$

The one double curve consists of the straight line $x = a$, $y = at$, $z = at$. Two points P_1 and P_2 on the generatrix g equally distant from M bound a

---

* Gino Loria, "Spezielle algebraische und transzendente Kurven," 2d ed., Vol. 1, pp. 358–369.

† This is the sixth species of the classification of quartic scrolls by Cayley, Collected Works, Vol. 6, p. 328, and the fifth species of that of Cremona in *Memorie della Accademia delle Scienze dell' Istituto di Bologna,* 2 Ser., Vol. 8, pp. 235–250 (1868).

segment which describes a unifacial band of Moebius.* In a similar manner, and in a more general sense, such a band is described on any surface of the class, and is unit or bifacial according as the order of the surface is odd or even. This surface is also discussed from a function theoretic standpoint by Weyl†.

4. *Quintic*, $p = 3$, $q = 2$

$$(y^3 - 3x^2y)z^2 + (2x^4 + 4x^2y^2 + 2y^4 - 2ax^3 + 6axy^2)z + (x^2 + y^2 - a^2)(y^3 - 3x^2y) = 0.$$

C_1 is a triple line of the surface, C_2 is single. There is one double curve of order 3 with the parametric equations

$$x = \frac{a(1 + t^2)}{1 - 3t^2}, \quad y = \frac{at(1 + t^2)}{1 - 3t^2}, \quad z = \frac{a(3t - t^3)}{1 - 3t^2}.$$

The projection of this curve upon the xy-plane has the polar equation

$$\rho = \frac{a}{\cos 3\theta},$$

and the cartesian equation

$$x^3 - 3xy^2 - a(x^2 + y^2) = 0.$$

This quintic has the index *AIII* in Schwarz's classification of quintic ruled surfaces, loc. cit., p. 57.

C. Number of Surfaces of a given Order.

When the order of the surface of the class is given, we may ask the question, how many species of the class are there with the same order? When the order is even, say $2n$, then $p + q = n$, and we may form all possible combinations of relative primes p and q with q odd, to satisfy this condition. For example when the order is 12, $p + q = 6$, and we have the possibilities (1) $p = 5$, $q = 1$; (2) $p = 2$, $q = 3$; (3) $p = 1$, $q = 5$, so that there are three surfaces of order 12. When the order is odd, say $p + q = m$, we may proceed in a similar manner; for example when $m = 13$. As q is now even we have the possibilities (1) $p = 11$, $q = 2$; (2) $p = 9$, $q = 4$; (3) $p = 7$, $q = 6$; (4) $p = 5$, $q = 8$; (5) $p = 3$, $q = 10$; (6) $p = 1$, $q = 12$. Hence there are six surfaces of the class of order 13.

In general, when the order is even, say $2n$ and n is even, then there are at most $n/2$ species of the same order; when n is odd, there at most $(n + 1)/2$. When the order is odd, say m, there are at most $(m - 1)/2$ species of the class with this order.

---

* Werke, Vol. 2, pp. 484–485 and pp. 519–521.

† "Die Idee der riemannschen Fläche," p. 26.

GEOMETRICAL SIGNIFICANCE OF ISOTHERMAL CONJUGACY OF A NET OF CURVES.

By E. J. Wilczynski.

Introduction.

Let

$$(1) \qquad Ddu^2 + 2D'dudv + D''dv^2$$

be the second fundamental differential form of a surface S, and let us consider a region R on this surface which is free from parabolic points so that, for all points in R,

$$(2) \qquad D'^2 - DD'' \neq 0.$$

If D' is equal to zero for all points of R, the curves $u =$ const. and $v =$ const. form a *conjugate* net. If this condition is satisfied, and if besides the ratio $D : D''$ assumes the form of a function of u alone multiplied by a function of v alone, so that

$$(3) \qquad \frac{\partial^2 \log D/D''}{\partial u \partial v} = 0, \qquad D' = 0,$$

the net is said to be *isothermally* conjugate. This name is due to Bianchi,* and was chosen by him because, in all such cases, it is possible to choose new variables

$$\bar{u} = \varphi(u), \qquad \bar{v} = \psi(v)$$

in such a way as to transform (1) into the isothermal form

$$\lambda(\bar{u}, \bar{v})(d\bar{u}^2 + d\bar{v}^2),$$

without changing the conjugate net under consideration.

Bianchi also proved that the property of isothermal conjugacy is of a *projective* character.† That is, if an isothermally conjugate net is subjected to any projective transformation, the resulting net will again be isothermally conjugate. But Bianchi did not furnish any geometric interpretation of the analytic conditions (3) which serve to define such systems. Moreover, although the importance of this notion was becoming more and more apparent, because of a steadily increasing body of theorems which made use of it, no serious attempt seems to have been made to discover its true significance until 1915, when the author of the present paper discovered an algebraic relation, between certain completely interpreted projective in-

* L. Bianchi, "Lezioni di geometria differenziale" (Seconda edizione), Vol. 1, p. 168.

† Ibid. p. 169.

variants, which is characteristic of isothermally conjugate systems.* Thus, in a sense, the problem was solved. But the solution was not altogether satisfying because it lacked simplicity and could not be formulated completely in terms of purely descriptive relations. A year afterward, the late G. M. Green, whose premature death has deprived geometry of one of its most brilliant students, took a long step in advance.† In fact, Green believed that he had settled the matter completely. But he had overlooked an important case in which his geometric criterion fails to distinguish between isothermally conjugate nets and nets of an entirely different kind.

The present paper was written for the purpose of completing the solution of this problem, as nearly as possible in the spirit of Green's method, and making use of Green's notations. I dedicate this paper to his memory.

1. Résumé and Revision of Green's Theory.

Let

$$y^{(k)} = y^{(k)}(u, v), \qquad (k = 1, 2, 3, 4) \tag{4}$$

be the homogeneous coördinates of a point P_y. When the variables, u and v, vary over their ranges, P_y will in general describe a surface S_y. We shall assume that this surface does not degenerate into a curve, and that it is non-developable. If the curves $u =$ const. and $v =$ const. form a conjugate net on S_y, there exists a completely integrable system of differential equations of the form

$$\begin{aligned} y_{uu} &= ay_{vv} + by_u + cy_v + dy, \qquad a \neq 0, \\ y_{uv} &= * + b'y_u + c'y_v + d'y, \end{aligned} \tag{5}$$

whose fundamental, linearly independent, solutions are $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$. Conversely, every completely integrable system of form (5) defines a non-developable surface referred to a conjugate net.

The integrability conditions of system (5) teach us that there exists a function p, of u and v, such that‡

$$p_u = b + 2c', \qquad p_v = \frac{2ab' - c - a_v}{a}. \tag{6}$$

Consequently we can make a transformation of the form

$$y = \lambda \bar{y}, \tag{7}$$

* E. J. Wilczynski, "The General Theory of Congruences," *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 323. Quoted hereafter as W.

† G. M. Green, "Projective Differential Geometry of One-parameter Families of Space Curves and Conjugate Nets on a Curved Surface (Second Memoir), Am. Jour. of Math., Vol. 38 (1916), p. 323. Quoted hereafter as Green (Second Memoir).

‡ G. M. Green, "Projective Differential Geometry of One-parameter Families of Space Curves, etc. (First Memoir), Am. Jour. of Math., Vol. 37 (1915), p. 223. Quoted hereafter as Green (First Memoir).

where λ is subjected to the conditions

$$\frac{\lambda_u}{\lambda} = \tfrac{1}{4}p_u, \qquad \frac{\lambda_v}{\lambda} = \tfrac{1}{4}p_v. \tag{8}$$

The resulting system of differential equations has the same form as (5), with the coefficients*

$$\begin{aligned} &A = a, \qquad B = b - \tfrac{1}{2}p_u, \qquad C = c + \frac{a}{2}p_v, \\ &D = d + \tfrac{1}{4}bp_u + \tfrac{1}{4}cp_v - \tfrac{1}{4}p_{uu} + \tfrac{1}{4}ap_{vv} - \tfrac{1}{16}p_u^2 + \tfrac{1}{16}ap_v^2, \\ &B' = b' - \tfrac{1}{4}p_v, \qquad C' = c' - \tfrac{1}{4}p_u, \\ &D' = d' + \tfrac{1}{4}b'p_u + \tfrac{1}{4}c'p_v - \tfrac{1}{4}p_{uv} - \tfrac{1}{16}p_up_v. \end{aligned} \tag{9}$$

These coefficients are *seminvariants* of (5), and the new system is said to be in its *canonical form*. The relations

$$B + 2C' = 0, \qquad 2AB' - C - A_v = 0, \tag{10}$$

which follow from (9), are characteristic of this canonical form.

Any proper transformation of the form

$$\bar{u} = \varphi(u), \qquad \bar{v} = \psi(v), \tag{11}$$

affects only the parametric representation of the conjugate net given by (5), but leaves the net itself unchanged. The *invariants* of the net are those functions of the seminvariants, which remain unchanged by transformations of form (11), except for a factor. The fundamental invariants are†

$$\begin{aligned} &\mathfrak{A} = A, \qquad \mathfrak{B}' = B' - \frac{3}{8}\frac{A_v}{A}, \qquad \mathfrak{C}' = C' + \frac{1}{8}\frac{A_u}{A}, \\ &\mathfrak{D}' = D' + B'C', \qquad \mathfrak{D} = D - (B'A_v - AB'_v) - C'_u + 3(AB'^2 - C'^2); \end{aligned} \tag{12}$$

besides these, the following two, the Laplace-Darboux invariants of the net,‡

$$H = D' + B'C' - B_u, \qquad K = D' + B'C' - C'_v \tag{13}$$

are especially important.

The curves $u =$ const. and $v =$ const. of our conjugate net are not asymptotic lines. Therefore, the osculating planes of the two curves of the net, which meet at a point P_y of the surface, determine, as their line of intersection, a line passing through P_y and not in the tangent plane. This line is called the *axis* of P_y, and the totality of all such lines is called the *axis congruence* of the given conjugate system. The developables of the

* Green (First Memoir), p. 224.

† Ibid., p. 226.

‡ Ibid., p. 231–232.

axis congruence correspond to a net of curves on S_y, called the *axis curves*, whose differential equation is*

$$(14)\quad a\left(K + 2b'_u - b_v - \frac{\partial^2 \log a}{\partial u \partial v}\right)du^2 - \mathfrak{D}du dv - (H + 2b'_u - b_v)dv^2 = 0,$$

where a, b, and b' may be replaced by A, B, B' and where the relations (10) may then be used. The *anti-axis curves* are defined by*

$$(15)\quad a\left(K + 2b'_u - b_v - \frac{\partial^2 \log a}{\partial u \partial v}\right)du^2 + \mathfrak{D}du dv - (H + 2b'_u - b_v)dv^2 = 0.$$

Their tangents at any point of the surface are the harmonic conjugates of the axis curve tangents with respect to the tangents of the original conjugate system $u =$ const., $v =$ const.

The covariants

$$(16)\qquad \rho = y_u - c'y, \qquad \sigma = y_v - b'y$$

are the variables which determine the Laplace transformations of system (5). The points P_ρ and P_σ are in the plane tangent to S_y at P_y. The locus of P_ρ is the second sheet of the focal surface of the congruence formed by the tangents of the curves $v =$ const. on S_y. P_σ is connected in the same way with the congruence of tangents of the curves $u =$ const. on S_y. The line $P\rho P\sigma$, which moreover corresponds to the axis of P_y by duality, is called the *ray of* P_y. The totality of rays, for all surface points, is called the *ray congruence*, and the curves on S_y which correspond to the developables of the ray congruence, are called its *ray curves.** The differential equation of the ray curves is

$$(17)\qquad aH du^2 - \mathfrak{D}du dv - K dv^2 = 0.$$

The *anti-ray curves* are related to the ray curves in the same way as the axis curves to the anti-axis curves. Their differential equation is as follows†;

$$(18)\qquad aH du^2 + \mathfrak{D}du dv - K dv^2 = 0.$$

There exists a uniquely determined conjugate net on the surface such that the two tangents of this new net, at any point of the surface, shall separate not only the asymptotic tangents, but also the tangents of the original conjugate system, harmonically. Green has called this system of curves the *associate conjugate net,*‡ and found its differential equation to be

$$(19)\qquad a du^2 - dv^2 = 0,$$

the asymptotic net of S_y being determined by

$$(20)\qquad a du^2 + dv^2 = 0.$$

* W., pp. 314–316 and Green (Second Memoir), pp. 308 and 310.
† W., pp. 317–318 and Green (Second Memoir), p. 309.
‡ Green (Second Memoir), p. 313.

In the case of an isothermally conjugate net, a has the form of a product of a function of u alone by a function of v alone, so that

$$\frac{\partial^2 \log a}{\partial u \partial v} = 0, \qquad a \neq 0. \tag{21}$$

It will then be possible to find a transformation

$$\bar{u} = U(u), \qquad \bar{v} = V(v),$$

such that the value of a in the transformed differential equations becomes equal to unity. Thus, *if the parametric net is isothermally conjugate, we may assume*

$$a = 1. \tag{22}$$

Let us consider the three quadratics (14), (18), and (19). The Jacobian of (14) and (19) is

$$a\mathfrak{D}du^2 + 2a\left(H - K + \frac{\partial^2 \log a}{\partial u \partial v}\right)dudv + \mathfrak{D}dv^2 = 0; \tag{23}$$

the Jacobian of (18) and (19) is

$$a\mathfrak{D}du^2 + 2a(H - K)dudv + \mathfrak{D}dv^2 = 0, \tag{24}$$

and clearly these Jacobians are equivalent, as quadratics in $du : dv$, if (21) is satisfied. But they are also equivalent if $\mathfrak{D} = 0$, and this is the case which Green failed to consider. In this exceptional case the axis curves and ray curves are so related to the parametric conjugate system that at every surface point the tangents belonging to the latter are separated harmonically by the tangents of each of the former nets, unless still other invariants vanishing cause one or both of these nets to become indeterminate. On account of these properties, let us call such conjugate nets, characterized by the condition $\mathfrak{D} = 0$, *harmonic conjugate nets.*

We have proved the following theorem.

THEOREM 1. *A conjugate net whose axis tangents, anti-ray tangents, and associate conjugate tangents, form three pairs of an involution at every point of the net, is either isothermally conjugate, or harmonic, or both.*

In this theorem, the axis tangents and anti-ray tangents may be replaced simultaneously by the anti-axis tangents and ray tangents, respectively.

Since it is our purpose to characterize isothermally conjugate nets completely by geometric properties, we must now search for properties of such nets which they do *not* share with harmonic conjugate nets. In most cases the following theorem will enable us to distinguish between harmonic and isothermally conjugate nets.

THEOREM 2. *The involution, mentioned in Theorem 1, has the parametric*

conjugate tangents as its double lines, if and only if the original net is harmonic. Therefore, the given net is isothermally conjugate, and not harmonic, if the three pairs of tangents mentioned in Theorem 1 are pairs of an involution, and if, besides, the double lines of this involution do not coincide with the parametric tangents.

If, however, the double elements of this involution *do* coincide with the parametric tangents, we can only conclude that the given net is harmonic. It may or may not be isothermally conjugate, at the same time. Thus our geometric criterion fails to distinguish between nets which are both isothermally conjugate and harmonic, and those which are merely harmonic.

Green* has shown that the associate conjugate net of an isothermally conjugate net is also isothermally conjugate, and vice versa, a theorem which we shall generalize in the next section. We may, therefore, apply theorems 1 and 2 to the associate conjugate net, obtaining the following result.

THEOREM 3. *If a conjugate net is isothermally conjugate, the associate conjugate net is also isothermally conjugate and vice versa. Consequently, the associate axis tangents, the associate anti-ray tangents, and the conjugate tangents of the original net, at any point of the net, will form three pairs of an involution. The double lines of this second involution will coincide with the associate conjugate tangents if and only if the associate conjugate net is harmonic.*

The associate axis tangents, etc., mentioned in this theorem, are related to the associate conjugate system in the same manner as the axis tangents, etc. are to the original system. By combining theorems 1, 2, 3, we obtain the following criterion.

THEOREM 4. *For an isothermally conjugate net both of the involutions, mentioned in theorems 1 and 3 exist. Conversely, if both of these involutions exist for a conjugate net, we can conclude that the net is isothermally conjugate unless both the original net and its associate net are harmonic.*

2. PENCILS OF CONJUGATE NETS ON A SURFACE.

Theorem 4 seems to be the most comprehensive criterion which can be obtained without introducing something essentially new into the discussion, but it does not solve the problem completely. For, it does not enable us to distinguish geometrically between conjugate nets which are harmonic, possess a harmonic associate net, and are besides isothermally conjugate, and conjugate nets which possess merely the first two of these properties. In order to solve our problem completely we introduce a new notion, that of a *pencil of conjugate systems,* a notion which we shall introduce at present only in connection with our special problem but which seems to be one of considerable general importance.

* Green (Second Memoir), p. 324.

Let us assume that the given conjugate system is isothermally conjugate, let the independent variables be chosen so that $a = 1$, and let the equations (5) be taken in their canonical form. Then we shall have

$$(25)\qquad \begin{aligned} y_{uu} &= y_{vv} + By_u + Cy_v + Dy, \qquad a = A = 1, \\ y_{uv} &= \ \ * \ + B'y_u + C'y_v + D'y, \end{aligned}$$

where, on account of (10),

$$(26)\qquad B = -2C', \qquad C = 2B'.$$

The differential equations of the original conjugate system will be $dudv = 0$, that of the associate system will be $du^2 - dv^2 = 0$, and that of the asymptotic lines will be $du^2 + dv^2 = 0$. The differential equation

$$(27)\qquad \alpha du^2 + 2\beta dudv + \gamma dv^2 = 0$$

will determine a conjugate net if and only if

$$\alpha + \gamma = 0,$$

a condition obtained by equating to zero the harmonic invariant of (27) and $du^2 + dv^2 = 0$, the differential equation of the asymptotic lines. The tangents, at any point, of the curves of such a conjugate net will divide the corresponding tangents of the original conjugate net in a *constant* cross-ratio, if and only if the ratio of α to β is a constant. Consequently, *the differential equation*

$$(28)\qquad (du + kdv)(kdu + dv) = 0,$$

where k is an arbitrary constant, will determine a one-parameter family of conjugate nets each of which has the property that, at every point, the two tangents which belong to it determine a constant cross-ratio with those which belong to the original net.

We shall speak of the one-parameter family of conjugate nets, determined in this way by a given one, as a *pencil of conjugate nets*. There is one such net for every value, real or complex, of the constant k, but it is clear from (28) that the same net will correspond to two values of k which are negative reciprocals of each other. The net which corresponds to $k = 0$ or $k = \infty$ is the original net, and that which corresponds to the values $k = \pm 1$ is the associate net. For $k = \pm i$ the two factors of (28) become identical with each other and with one of the factors of $du^2 + dv^2$; the net degenerates into one of the families of asymptotic lines counted twice and therefore is not, properly speaking, a net at all. By a *proper* net of the pencil we mean any one of its nets excepting the two just mentioned which correspond to $k = i$ and $k = -i$. Of course every proper net of a pencil may be regarded as determining, in its turn, a pencil of nets.

But it follows at once, from the definition of a pencil, that all of these pencils coincide with each other and with the original pencil, and further that the nets of a pencil may be arranged in pairs associate to each other.

In order to study the properties of an individual net of the pencil, we introduce the variabels

$$\bar{u} = u - kv, \qquad \bar{v} = ku + v \tag{29}$$

into (25) in place of u and v. These variables will be independent if $1 + k^2$ is different from zero. We shall assume

$$1 + k^2 \neq 0, \tag{30}$$

a hypothesis which excludes from consideration only the improper conjugate systems formed by each of the two sets of asymptotic lines. We find

$$\begin{aligned} y_u &= y_{\bar{u}} + k y_{\bar{v}}, \qquad y_v = -k y_{\bar{u}} + y_{\bar{v}}, \\ y_{uu} &= y_{\bar{u}\bar{u}} + 2k y_{\bar{u}\bar{v}} + k^2 y_{\bar{v}\bar{v}}, \\ y_{uv} &= -k y_{\bar{u}\bar{u}} + (1 - k^2) y_{\bar{u}\bar{v}} + k y_{\bar{v}\bar{v}}, \\ y_{vv} &= k^2 y_{\bar{u}\bar{u}} - 2k y_{\bar{u}\bar{v}} + y_{\bar{v}\bar{v}}. \end{aligned} \tag{31}$$

Substitution of these values into equations (25) gives

$$\begin{aligned} (1 - k^2) y_{\bar{u}\bar{u}} + 4k y_{\bar{u}\bar{v}} - (1 - k^2) y_{\bar{v}\bar{v}} &= B(y_{\bar{u}} + k y_{\bar{v}}) + C(-k y_{\bar{u}} + y_{\bar{v}}) + Dy, \\ -k y_{\bar{u}\bar{u}} + (1 - k^2) y_{\bar{u}\bar{v}} + k y_{\bar{v}\bar{v}} &= B'(y_{\bar{u}} + k y_{\bar{v}}) + C'(-k y_{\bar{u}} + y_{\bar{v}}) + D'y, \end{aligned}$$

whence

$$\begin{aligned} (1 + k^2)^2 (y_{\bar{u}\bar{u}} - y_{\bar{v}\bar{v}}) &= (1 - k^2)[(B - kC) y_{\bar{u}} + (kB + C) y_{\bar{v}} + Dy] \\ &\quad - 4k[(B' - kC') y_{\bar{u}} + (kB' + C') y_{\bar{v}} + D'y], \\ (1 + k^2)^2 y_{\bar{u}\bar{v}} &= k[(B - kC) y_{\bar{u}} + (kB + C) y_{\bar{v}} + Dy] \\ &\quad + (1 - k^2)[(B' - kC') y_{\bar{u}} + (kB' + C') y_{\bar{v}} + D'y]. \end{aligned} \tag{32}$$

These equations show, in the first place, that the new conjugate net, $\bar{u} =$ const., $\bar{v} =$ const., is isothermally conjugate, giving

THEOREM 5. *An isothermally conjugate net determines a pencil, all of whose proper nets are isothermally conjugate.*

This theorem includes, as a special case, Green's theorem that the associate net of an isothermally conjugate net is also isothermally conjugate. But we may draw a still farther reaching conclusion, by remembering that the same pencil of nets is determined if we start from any one of its proper nets in place of the one actually used. We then obtain the following result.

THEOREM 6. *If a pencil of conjugate nets contains one isothermally conjugate net, then all proper nets of the pencil are isothermally conjugate.*

We may reduce (32) to the form (25) by dividing by $(1 + k^2)^2$. If we denote the corresponding coefficients by A_k, B_k, C_k, etc., we find

$$
(33)\qquad
\begin{aligned}
(1+k^2)^2B_k &= (1-k^2)(B-kC) - 4k(B'-kC'),\\
(1+k^2)^2C_k &= (1-k^2)(kB+C) - 4k(kB'+C'),\\
(1+k^2)^2D_k &= (1-k^2)D - 4kD',\\
(1+k^2)^2B'_k &= k(B-kC) + (1-k^2)(B'-kC'),\\
(1+k^2)^2C'_k &= k(kB+C) + (1-k^2)(kB'+C'),\\
(1+k^2)^2D'_k &= kD + (1-k^2)D',
\end{aligned}
$$

and

$$
(34)\qquad A_k = 1, \qquad B_k = -2C'_k, \qquad C_k = 2B'_k.
$$

The relation $A_k = 1$ is equivalent to Theorem 5. The other two relations in (34) may be verified by means of (26) and (33). They show that our transformed system of differential equations is in its canonical form.

The invariant $\mathfrak{D}$, whose vanishing characterizes the original conjugate system as a harmonic one, reduces to

$$
(35)\qquad \mathfrak{D} = D + B'_v - C'_u + 3(B'^2 - C'^2),
$$

since we are assuming $A = 1$. Let us denote by $\mathfrak{D}_k$ the corresponding invariant for any conjugate system of the pencil, so that

$$
(36)\qquad \mathfrak{D}_k = D_k + (B'_k)_{\bar v} - (C'_k)_{\bar u} + 3(B'_k)^2 - 3(C'_k)^2.
$$

From (33) and (26) we find

$$
(37)\qquad
\begin{aligned}
(1+k^2)^2B'_k &= (1-3k^2)B' + (k^3-3k)C',\\
(1+k^2)^2C'_k &= -(k^3-3k)B' + (1-3k^2)C',\\
(1+k^2)^2D_k &= (1-k^2)D - 4kD'.
\end{aligned}
$$

If θ is any function of u and v, we find from (31),

$$
(38)\qquad \theta_{\bar u} = \frac{1}{1+k^2}(\theta_u - k\theta_v), \qquad \theta_{\bar v} = \frac{1}{1+k^2}(k\theta_u + \theta_v).
$$

Consequently we obtain the formulæ

$$
\begin{aligned}
(1+k^2)^3(C'_k)_{\bar u} &= -(k^3-3k)(B'_u - kB'_v) + (1-3k^2)(C'_u - kC'_v),\\
(1+k^2)^3(B'_k)_{\bar v} &= (1-3k^2)(kB'_u + B'_v) + (k^3-3k)(kC'_u + C'_v),
\end{aligned}
$$

whence

$$
(39)\qquad
\begin{aligned}
(1+k^2)^4\mathfrak{D}_k = (1+k^2)^2[(1-k^2)D &- 4kD' - 2k(B'_u + C'_v)\\
&+ (1-k^2)(B'_v - C'_u)]\\
+ 3(1-k^2)(1-14k^2+k^4)&(B'^2 - C'^2)\\
&+ 12k(1-3k^2)(k^2-3)B'C'.
\end{aligned}
$$

For $k = 0$, $\mathfrak{D}_k$ reduces to (35), and for $k = 1$ to

$$
(40)\qquad \mathfrak{D}_1 = -D' - \tfrac{1}{2}(B'_u + C'_v) + 3B'C'.
$$

Let us assume that $\mathfrak{D}$ and $\mathfrak{D}_1$ are both equal to zero, so that both the original net and its associate are harmonic, besides being isothermally conjugate. Then $\mathfrak{D}_k$ reduces to the value given by

(41) $(1+k^2)^4\mathfrak{D}_k = -48k(1-k^2)[k(B'^2 - C'^2) + (1-k^2)B'C']$.

If the ratio $B' : C'$ is not a constant, $\mathfrak{D}_k$ can not be equal to zero, for all values of u and v, unless either $k = 0$ or $k = \pm 1$, and these values of k correspond to the original conjugate system and its associate. If the ratio $B' : C'$ *is* a constant which is finite, different from zero or unity, we obtain two values of k, negative reciprocals of each other, and different from 0, ∞, $+1$, or -1, by equating to zero the bracketed expression in (41). Thus, there may exist a third net of the pencil, besides the original net and its associate, for which $\mathfrak{D}_k$ is equal to zero. But if $\mathfrak{D}_k$ is equal to zero for more than three distinct nets of the pencil, $\mathfrak{D}_k$ will be equal to zero for all values of k, and B' and C' must vanish. In this case the differential equations of the net reduce to

$$y_{uu} = y_{vv}, \qquad y_{uv} = 0. \tag{42}$$

Nets of this sort may be described in very simple terms. From equations (42), we conclude

$$y = U(u) + V(v), \qquad U'' = V'' = \tfrac{1}{2}a_1,$$

where $U(u)$ and $V(v)$ are functions of the single variables indicated, and where a_1 is an arbitrary constant. But these equations furnish the following completely integrated expression for y;

$$y = a_1(u^2 + v^2) + a_2 u + a_3 v + a_4,$$

where a_1, a_2, a_3, a_4 are arbitrary constants. The homogeneous parametric equations of such a net may, therefore, be written in the form

$$y_1 = u^2 + v^2, \qquad y_2 = u, \qquad y_3 = v, \qquad y_4 = 1,$$

whence

$$y_1 y_4 - y_2^2 - y_3^2 = 0, \qquad y_2 - u y_4 = 0, \qquad y_3 - v y_4 = 0.$$

Therefore *the sustaining surface of such a net is a quadric. Each of the two component one-parameter families of the net is composed of plane curves (conics), whose planes form a pencil. The axes of these two pencils are conjugate tangents of the quadric surface at one of its points.*

A net with these properties shall be called an *isothermally conjugate quadratic net.* Making use of this terminology we have the following result.

THEOREM 7. *A pencil of isothermally conjugate nets which contains more than three distinct proper harmonic nets is composed entirely of isothermally conjugate quadratic nets.*

We are now in a position to obtain a geometric test for isothermal conjugacy which will be effective in those cases in which theorems 1–4 do not suffice. If a net is isothermally conjugate, every net of its pencil has the property described in theorem 1. If besides, more than three, and

therefore all, of these nets are harmonic, it is an isothermally conjugate quadratic net. Leaving aside this case, we see that the isothermal conjugacy of a net is assured if the property of theorem 1 holds for all of the nets of the pencil and if besides at least one of these nets is known to be non-harmonic.

We may formulate our resulting criterion in the following two theorems.

THEOREM 8. *An isothermally conjugate net possesses the following properties. At every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Moreover, all of the conjugate nets of the pencil, which is determined by the original net, possess this same property, and no more than three of these nets will be, at the same time, harmonic except in the case of an isothermally conjugate quadratic net.*

THEOREM 9. *Conversely: let there be given a conjugate net such that, at every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Let all of the conjugate nets of the pencil, determined by the given net, possess the same property, and assume that at least one of the nets of this pencil is not harmonic. Then the original net is isothermally conjugate. If, however, all of the nets of the pencil are harmonic, the original net is an isothermally conjugate net, if and only if it is an isothermally conjugate quadratic net.*

Theorems 8 and 9 together constitute a set of necessary and sufficient conditions for isothermal conjugacy, and these conditions are expressed in purely geometric form. For, according to theorem 2, the question whether a conjugate net is, or is not harmonic, may be decided by examining the double lines of the corresponding involution.

THE UNIVERSITY OF CHICAGO,
May 4, 1920.

NOTE ADDED OCTOBER 6, 1920.

This criterion may be simplified. I have found recently that, if all of the conjugate nets of a pencil are harmonic, they must also be isothermally conjugate. This remark enables us to replace Theorem 9 by

THEOREM 10. *Conversely, let there be given a conjugate net such that, at every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Let all of the conjugate nets of the pencil, determined by the given net, possess the same property. Then the original net is isothermally conjugate.*

I have also found a second characteristic property of isothermally conjugate nets, which admits of a far simpler statement than that described in theorems 8 and 10. But the detailed presentation of these matters must be left for a future occasion.

OBSERVATIONS WEIGHTED ACCORDING TO ORDER.

By P. J. Daniell.

1. *Introduction.*—When a series of measurements of some quantity are made, two particular quantities require to be calculated expressing respectively the norm and the deviation. For the norm the mean or the median is used while there are three measures of dispersion, the standard or root-mean-square deviation, the mean numerical deviation and the quartile deviation. The question is as to which of these are the more accurate under a general law. Moreover if we choose for our norm the mean or average it appears occasionally profitable to discard one or several extreme measures. Whether, or in what cases, this is legitimate is discussed by Poincaré* but no general conclusions are obtained.

Besides such a discard-average we might invent others in which weights might be assigned to the measures according to their order. In fact the ordinary average or mean, the median, the discard-average, the numerical deviation (from the median, which makes it minimum), and the quartile deviation can all be regarded as calculated by a process in which the measures are multiplied by factors which are functions of order. It is the general purpose of this paper to obtain a formula for the mean square deviation of any such expression. This formula may then be used to measure the relative accuracies of all such expressions.

Certain particular types are discussed and their accuracies calculated in percentages.

Unfortunately the standard deviation is not of the same general type and therefore we add a note on its accuracy. The assumptions made are fairly general. On the one hand the number of observations, n, is supposed large and terms of order higher than $1/n$ are discarded; on the other the probability law assumed is regular and indefinitely differentiable. In our applications to special types, however, we shall only consider cases in which the theoretical distribution is symmetrical, and this for logical reasons. It is useless to compare the relative merits of the various kinds of average, for example, the mean and the median, unless they all tend to coincide when n increases indefinitely. If there is a lack of symmetry both the mean and the median are necessary, or at least valuable, indications of the nature of the distribution. Indeed, in practise, their difference is sometimes regarded as a measure of lack of symmetry.

* Poincaré, "Calcul des Probabilités" (1912), p. 211.

2. *Mathematical Analysis.*—Assume that n measurements $t_1, t_2, \cdots, t_n$ are made and that their magnitudes are in the order of their suffixes, so that

$$t_1 \leqq t_2, \text{ and so on.}$$

Multiply by the factors $f_1, f_2, \cdots, f_n$, so that

$$\bar{t} = \sum_{r=1}^{n} f_r t_r.$$

We desire to find a formula for the mean square deviation of $\bar{t}$ when the measurements, t_r, are subject to some law of probability $p(t)$.

If $\varphi(t_1, \cdots, t_n)$ is some function of the measures considered in their proper order, the average value of φ when $t_1, \cdots, t_n$ vary according to the law of probability will be denoted by Av (φ) to distinguish this from the weighted average, $\bar{t}$, which we obtain for a particular fixed set of values $t_1, t_2, \cdots, t_n$.

Allowing for the possible permutations of the suffixes,

$$\text{Av}\ (\varphi) = n! \int_{-\infty}^{+\infty} p(t_n)dt_n \int_{-\infty}^{t_n} p(t_{n-1})dt_{n-1} \cdots \int_{-\infty}^{t_2} p(t_1)\varphi(t_1, \cdots, t_n)dt_1.$$

If

$$\int_{-\infty}^{t} p(t)dt = x,$$

let $t = t(x)$; then x varies from 0 to 1, and

$$\text{Av}\ (\varphi) = n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} \psi(x_1, \cdots, x_n)dx_1, \qquad (1)$$

where

$$\psi(x_1, \cdots, x_n) = \varphi[t(x_1), \cdots, t(x_n)].$$

We shall make frequent use of the formula

$$\int_0^a dx_p \int_0^{x_p} dx_{p-1} \cdots \int_0^{x_1} f(x)dx = \frac{1}{p!}\int_0^a f(x)(a - x)^p dx. \qquad (2)$$

This formula can readily be verified by differentiating with respect to a. A particular case is that in which $f(x) = 1$,

$$\int_0^a dx_p \int_0^{x_p} dx_{p-1} \cdots \int_0^{x_1} dx = \frac{1}{p!}\int_0^a (a - x)^p dx = \frac{1}{(p+1)!}a^{p+1}. \qquad (2a)$$

Substituting from (2a) in (1)

$$\text{Av}\ (1) = n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 = n! \cdot \frac{1}{n!} \cdot 1^n = 1.$$

This confirms the coefficient $n!$ in the formula (1).

$$\begin{aligned}\text{Av}\,(t_r) &= n!\int_0^1 dx_n \int_0^{x_n} dx_{n-1}\cdots\int_0^{x_2} t(x_r)dx_1 \\ &= n!\int_0^1 dx_n \int_0^{x_n} dx_{n-1}\cdots\int_0^{x_{r+1}} t(x_r)\frac{x_r^{r-1}}{(r-1)!}dx_r \quad \text{[by 2}a\text{]} \\ &= \frac{n!}{(n-r)!\,(r-1)!}\int_0^1 t(x)(1-x)^{n-r}x^{r-1}dx. \qquad (3)\end{aligned}$$

When r, n are large the integrand will have a steep maximum near $x = r/n$. Also

$$\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-r}x^{r-1}dx = \frac{r}{n+1}\cdot\frac{r+1}{n+2}\cdots\frac{r+p-1}{n+p}:$$

Denote $r/(n+1)$ by x_r and neglect terms of order higher than $1/n$.

$$\frac{r+1}{n+2} = x_r + \frac{1}{n}(1-x_r), \qquad \frac{r+2}{n+3} = x_r + \frac{2}{n}(1-x_r), \qquad \text{etc.}$$

$$\begin{aligned}&\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-r}x^{r-1}dx \\ &\qquad = x_r\left[x_r + \frac{1}{n}(1-x_r)\right]\cdots\left[x_r + \frac{p-1}{n}(1-x_r)\right] \\ &\qquad = x_r{}^p + \frac{p(p-1)}{2n}x_r{}^{p-1}(1-x_r).\end{aligned}$$

$$\begin{aligned}&\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 (x-x_r)^p(1-x)^{n-r}x^{r-1}dx \\ &\qquad = \sum_{q=0}^{p}\frac{p!}{q!(p-q)!}(-1)^q x_r^p \\ &\qquad\quad + \sum_{q=0}^{p}\frac{p!}{q!(p-q)!}(-1)^q\frac{(p-q)(p-q-1)}{2n}x_r^{p-1}(1-x_r).\end{aligned}$$

Of these two sums the former is 0 unless $p = 0$ and the latter is 0 unless $p = 2$.

$$\begin{aligned}\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 (x-x_r)^p(1-x)^{n-r}x^{r-1}dx &= 0 \qquad (p \neq 0, 2) \\ &= 1 \qquad (p = 0) \\ &= \frac{1}{n}x_r(1-x_r) \qquad (p = 2).\end{aligned}$$

[The reader is reminded that these equations are satisfied only as far as terms of order $1/n$.]

Expand $t(x)$ by a Taylor development near $x = x_r$,

$$t(x) = t(x_r) + (x - x_r)t'(x_r) + \frac{(x - x_r)^2}{2!} t''(x_r) + \cdots.$$

Substitute into (3) and use the formula just obtained, then

$$\text{Av}\,(t_r) = t(x_r) + \frac{1}{2n} x_r(1 - x_r)t''(x_r). \tag{4}$$

By the same reasoning,

$$\begin{aligned} \text{Av}\,(t_r^2) &= t^2(x_r) + \frac{1}{2n} x_r(1 - x_r)[2t(x_r)t''(x_r) + 2\{t'(x_r)\}^2] \\ &= [\text{Av}\,(t_r)]^2 + \frac{1}{n} x_r(1 - x_r)[t'(x_r)]^2. \end{aligned} \tag{5}$$

We next require to calculate Av (t_rt_s) and must agree on order. Suppose $s > r$, then

$$\begin{aligned} \text{Av}\,(t_rt_s) &= n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} t(x_r)t(x_s)dx_1 \\ &= \frac{n!}{(s - r - 1)!\,(r - 1)!} \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \\ &\qquad \cdots \int_0^{x_{s+1}} t(x)dx \int_0^x t(y)(x - y)^{s-r-1}y^{r-1}dy \qquad (6) \\ &= \frac{n!}{(n - s)!\,(s - r - 1)!\,(r - 1)!} \\ &\qquad \times \int_0^1 (1 - x)^{n-s}t(x)dx \int_0^x (x - y)^{s-r-1}y^{r-1}t(y)dy. \end{aligned}$$

In this double integral the integrand has a steep maximum near

$$x = s/n, \qquad y = r/n.$$

$$\begin{aligned} &\frac{n!}{(n - s)!\,(s - r - 1)!\,(r - 1)!} \int_0^1 x^p(1 - x)^{n-s}dx \int_0^x y^q(x - y)^{s-r-1}y^{r-1}dy \\ &= \frac{r}{n + 1} \cdot \frac{r + 1}{n + 2} \cdots \frac{r + q - 1}{n + q} \cdot \frac{s + q}{n + q + 1} \cdots \frac{s + q + p - 1}{n + q + p}. \end{aligned}$$

Denote $r/(n + 1)$ by x_r, $s/(n + 1)$ by x_s and expand as far as $1/n$,

$$\frac{r + 1}{n + 2} = x_r + \frac{1}{n}(1 - x_r), \qquad \text{etc.},$$

$$\frac{s + q}{n + q + 1} = x_s + \frac{q}{n}(1 - x_r), \qquad \text{etc.}$$

$$\frac{n!}{(n-s)!\,(s-r-1)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-s}dx\int_0^x y^q(x-y)^{s-r-1}y^{r-1}dy$$

$$= x_r^q x_s^p + \frac{q(q-1)}{2n}x_r^{q-1}(1-x_r)x_s^p + \frac{p(2q+p-1)}{2n}x_r^q x_s^{p-1}(1-x_s)$$

$$= x_r^q x_s^p + \frac{q(q-1)}{2n}x_r^{q-1}(1-x_r)x_s^p + \frac{p(p-1)}{2n}x_s^{p-1}(1-x_s)x_r^q$$

$$+\frac{pq}{n}x_r^{q-1}x_s^{p-1}x_r(1-x_s).$$

Using a method similar to that given above,

$$\frac{n!}{(n-s)!\,(s-r-1)!\,(r-1)!}$$

$$\times\int_0^1 (x-x_s)^p(1-x)^{n-s}dx\int_0^x (y-x_r)^q(x-y)^{s-r-1}y^{r-1}dy = 0,$$

except for $p = 0,\ q = 0;\ p = 0,\ q = 2;\ p = 2;\ q = 0;\ p = 1,\ q = 1.$

In formula (6) expand

$$t(x) = t(x_s) + (x-x_s)t'(x_s) + \cdots,\qquad t(y) = t(x_r) + (y-x_r)t'(x_r) + \cdots,$$

then by similar reasoning as before

$$\begin{aligned}\text{Av}\ (t_rt_s),\ (s>r) &= t(x_r)t(x_s) + \frac{1}{2n}x_r(1-x_r)t''(x_r)t(x_s)\\ &\quad + \frac{1}{2n}x_s(1-x_s)t(x_r)t''(x_s) + \frac{1}{n}x_r(1-x_s)t'(x_r)t'(x_s) \qquad (7)\\ &= \text{Av}\ (t_r)\cdot\text{Av}\ (t_s) + \frac{1}{n}x_r(1-x_s)t'(x_r)t'(x_s).\end{aligned}$$

Now

$$\bar t = \sum_{r=1}^n f_rt_r$$

$$\text{Av}\ (\bar t) = \sum_{r=1}^n f_r\ \text{Av}\ (t_r),$$

$$\begin{aligned}\text{Av}\ (\bar t^2) &= \sum_{r=1}^n f_r^2\ \text{Av}\ (t_r^2) + 2\sum_{r=1}^n\ \sum_{s=r+1}^n f_rf_s\ \text{Av}\ (t_rt_s) \qquad [\text{By } 5,\ 7].\\ &= \sum_{r=1}^n f_r^2[\text{Av}\ (t_r)]^2 + 2\sum_{r=1}^n\ \sum_{s=r+1}^n f_rf_r\ \text{Av}\ (t_r)\cdot\text{Av}\ (t_s)\\ &\quad + \sum_{r=1}^n f_r^2\frac{x_r(1-x_r)}{n}[t'(x_r)]^2\\ &\quad + 2\sum_{r=1}^n\ \sum_{s=r+1}^n f_rf_r\frac{x_r(1-x_s)}{n}t'(x_r)t'(x_s)\end{aligned}$$

Let $S^2 = n \times x$ mean square deviation of $(\bar{t}) = n[\mathrm{Av}\,(\bar{t^2}) - \mathrm{Av}^2\,(\bar{t})]$. Then let

$$f(x_r) = nf_r, \qquad f(x_r)\Delta x = f(x_r) \cdot \frac{1}{n} = f_r,$$

and replace the double sum by an integral.

$$S^2 = 2\int_0^1 f(x)(1-x)t'(x)dx \int_0^x f(y)yt'(y)dy.$$

Let

$$\varphi(t) = \int_c^t f[x(t)]dt,$$

where c is so chosen that

$$\int_{-\infty}^{+\infty} \varphi(t)p(t)dt = 0. \tag{8}$$

Let

$$\psi(x) = \varphi[t(x)], \qquad \psi'(x) = f(x)t'(x).$$

$$S^2 = 2\int_0^1 (1-x)\psi'(x)dx \int_0^x y\psi'(y)dy.$$

Consider the function

$$F(a, b) = 2\int_a^b (b-x)\psi'(x)dx \int_a^x (y-a)\psi'(y)dy,$$

$$\frac{\partial F}{\partial b} = 2\int_a^b \psi'(x)dx \int_a^x (y-a)\psi'(y)dy,$$

$$\frac{\partial^2 F}{\partial a \partial b} = -2\int_a^b \psi'(x)dx \int_a^x \psi'(y)dy = -[\psi(b) - \psi(a)]^2.$$

Integrating again and since $\partial F/\partial b = 0$, $F = 0$ when $a = b$,

$$\frac{\partial F}{\partial b} = \int_a^b [\psi(b) - \psi(y)]^2 dy, \qquad \frac{dF(0, b)}{db} = \int_0^b [\psi(b) - \psi(y)]^2 dy,$$

$$F(0, b) = \int_0^b dx \int_0^x dy[\psi(x) - \psi(y)]^2,$$

$$\begin{aligned} S^2 &= F(0, 1) \\ &= \int_0^1 dx \int_0^x dy[\psi(x) - \psi(y)]^2 \\ &= \int_{-\infty}^{+\infty} p(t)dt \int_{-\infty}^t p(u)du[\varphi(t) - \varphi(u)]^2. \end{aligned}$$

Interchange the order of integration and also the symbols t, u.

$$S^2 = \int_{-\infty}^{+\infty} p(t)dt \int_t^{\infty} p(u)du[\varphi(u) - \varphi(t)]^2.$$

Combining both forms,

$$S^2 = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[\varphi(t) - \varphi(u)]^2 p(t)p(u)dtdu$$

$$= \int_{-\infty}^{+\infty}\varphi^2(t)p(t)dt - \left[\int_{-\infty}^{+\infty}\varphi(t)p(t)dt\right]^2.$$

But by (8) the last term is 0; then

$$S^2 = \int_{-\infty}^{+\infty}\varphi^2(t)p(t)dt. \tag{9}$$

This is the formula we set out to obtain.

3. *Norm and Deviation.*—For the norm or average $\bar{t} = \sum_r f_r t_r$, with the condition $\sum_{r=1}^{n} f_r = 1$.

Expressing this by the approximate integral and then integrating by parts,

$$\int_{-\infty}^{+\infty} f(t)p(t)dt = 1,$$

$$\int_{-\infty}^{+\infty}[-p'(t)]\varphi(t)dt = 1. \tag{10}$$

The mean is obtained by equal weighting, $f(t) = 1$, $\varphi(t) = t - t_0$, where t_0 is the theoretical average. Then

$$S^2 = \int_{-\infty}^{+\infty}(t - t_0)^2 p(t)dt = \sigma^2.$$

Then the mean square deviation of the mean of n measurements is

$$S^2/n = \sigma^2/n.$$

This particular result is well-known but it confirms our formula (9).

If several groups of measures are to be combined, the average from each group should be multiplied by a factor inversely as the square of the deviation in that group. If then we agree to take the accuracy of the mean as a standard, equal to 1 or 100 per cent., the accuracy of any norm will be measured by the ratio σ^2/S^2.

Definition.—The accuracy of a norm is defined to be σ^2/S^2, where σ^2 is the theoretical square deviation.

In the case of the measure of deviation condition (10) no longer applies but we must suppose the weights f_r chosen so that the average value of the deviation has a fixed value, D.

$$\text{Av}\,(\bar{t}) = \sum_{r=1}^{n} f_r\,\text{Av}\,(t_r) = D.$$

Expressing in integral form, and integrating by parts,

$$D = \int_{-\infty}^{+\infty} tf(t)p(t)dt$$

$$= \int_{-\infty}^{+\infty} \frac{d}{dt}(-tp)\varphi(t)dt \qquad \text{[By (4)]}$$

$$= \int_{-\infty}^{+\infty} [-tp'(t)]\varphi(t)dt - \int_{-\infty}^{+\infty} p(t)\varphi(t)dt.$$

Then, by (8),

$$\int_{-\infty}^{+\infty} [-tp'(t)]\varphi(t)dt = D. \tag{11}$$

For the measure of deviation condition (11) takes the place of (10).

Again if we double the value of D, by doubling f_r, we shall multiply S^2 by 4. A true measure of accuracy will be some multiple of D^2/S^2, and for reasons which appear later we make the

Definition.—The accuracy of a measure of deviation is defined to be $D^2/(2S^2)$, where D is the theoretical average deviation.

Standard Deviation.—The standard deviation may be defined as D where

$$D^2 = \frac{1}{n}\sum_{r=1}^{n} t_r^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} t_r\right)^2.$$

It is difficult if not impossible to obtain a formula, in the general case, for the average value of D; nevertheless, if the number n is large, the proportional error in D will be small of order $1/\sqrt{n}$. We have the right, therefore, to assume that the proportional error in D will be one half that in D^2. Then if

$$D' = D^2, \qquad S'^2 = n \times \text{mean square deviation of } D',$$

$$\frac{D^2}{2S^2} = \frac{2D'^2}{S'^2}.$$

Choose the origin for t so that

$$\text{Av}\,(t) = \int_{-\infty}^{+\infty} tp(t)dt = 0.$$

Let

$$\int_{-\infty}^{+\infty} t^2p(t)dt = \sigma^2, \qquad \int_{-\infty}^{+\infty} t^4p(t)dt = q^4.$$

Then

$$\text{Av}\,(t_r) = \int t_rp(t_r)dt_r \int^{(n+1)} \cdots \int p(t_1)\cdots p(t_n)dt_1\cdots dt_n$$

$$= \int tp(t)dt = 0.$$

$$\text{Av}\,(t_rt_s) = \text{Av}\,(t_rt_s^3) = 0, \qquad \text{Av}\,(t_r^2) = \sigma^2,$$

$$\text{Av}\,(t_r^4) = q^4, \qquad \text{Av}\,(t_r^2t_s^2) = \sigma^2\cdot\sigma^2 = \sigma^4.$$

But

$$D' = \frac{1}{n}\sum_{r=1}^{n} t_r^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} t_r\right)^2.$$

The only terms in D' and D'^2 which yield integrals different from 0 will be of the types t_r^2, $t_r^2 t_s^2$, t_r^4.

$$\text{Av}\,(D') = \frac{1}{n}(n\sigma^2) - \frac{1}{n^2}(n\sigma^2) = \frac{n-1}{n}\sigma^2.$$

$$\begin{aligned}\text{Av}\,(D'^2) &= \frac{1}{n^2}\left[nq^4 + 2\frac{n(n-1)}{2}\sigma^4\right] - \frac{2}{n^3}\left[nq^4 + 2\frac{n(n-1)}{2}\sigma^4\right] \\ &\qquad + \frac{1}{n^4}\left[nq^4 + 6\frac{n(n-1)}{2}\sigma^4\right] \\ &= \frac{(n-1)^2}{n^3}q^4 + \frac{n-1}{n^3}(n^2 - 2n + 3)\sigma^4.\end{aligned}$$

$$S'^2 = n[\text{Av}\,(D'^2) - \{\text{Av}\,(D')\}^2] = \left(\frac{n-1}{n}\right)^2 q^4 - \frac{(n-1)(n-3)}{n^2}\sigma^4.$$

Omitting terms in $1/n$, $1/n^2$,

$$S^2 = \frac{1}{4}D^2 S'^2 \div D'^2 = \frac{D^2}{4}\frac{q^4 - \sigma^4}{\sigma^4}. \tag{12}$$

This formula gives the value of n times the mean square deviation of the standard deviation D.

When the theoretical distribution is normal or Gaussian,

$$q^4 = 3\sigma^4, \qquad S^2 = \frac{D^2}{4}\frac{3-1}{1} = \frac{D^2}{2}.$$

By the definition, the accuracy will be

$$\frac{D^2}{2S^2} = 1 = 100 \text{ per cent.}$$

This explains the factor 2 which is introduced to make the accuracy of the standard deviation 1 when the law is normal.

It is an interesting fact that the formula (12) proved for the standard deviation is the same as the corresponding value given by (9), when, instead of the mean-root-square deviation, we multiply every measurement by the theoretical value of t corresponding to its order in the series. For then

$$f(t) = \lambda t, \qquad \varphi(t) = \tfrac{1}{2}\lambda t^2 + \text{constant}.$$

This constant is chosen so that $\text{Av}\,(\varphi) = 0$, or

$$\varphi(t) = \tfrac{1}{2}\lambda(t^2 - \sigma^2).$$

Using the condition which led up to (11),

$$\lambda\sigma^2 = D.$$

From (9)

$$S^2 = \frac{\lambda^2}{4}\int_{-\infty}^{+\infty}(t^2 - \sigma^2)^2 p(t)\,dt$$

$$= \frac{\lambda^2}{4}(q^4 - 2\sigma^4 + \sigma^4) = \frac{D^2}{4}\frac{q^4 - \sigma^4}{\sigma^4}.$$

4. *Most Accurate Weighting.*—For the norm it is required to make S^2 given by formula (9) a minimum under condition (10). Then

$$\varphi_1(t) = \lambda_1\left[-\frac{p'(t)}{p(t)}\right], \qquad f_1(t) = \lambda_1\frac{d}{dt}\left[-\frac{p'(t)}{p(t)}\right],$$

where from (10)

$$\frac{1}{\lambda_1} = \int_{-\infty}^{+\infty}\frac{p'^2}{p}\,dt, \qquad S^2 = \lambda_1.$$

Accuracy

$$A_1 = \frac{\sigma^2}{S^2} = \sigma^2\int_{-\infty}^{+\infty}\frac{p'^2}{p}\,dt.$$

For the deviation, S^2 given by (9) is minimum under condition (11).

$$\varphi_2(t) = \lambda_2\left(-t\frac{p'}{p} - 1\right), \qquad f_2(t) = \lambda_2\frac{d}{dt}\left(-t\frac{p'}{p}\right),$$

where from (11)

$$\frac{D}{\lambda_2} = \int_{-\infty}^{+\infty} t^2\frac{p'^2}{p}\,dt - 1, \qquad S^2 = \lambda_2 D = D^2\frac{\lambda_2}{D}.$$

Accuracy

$$A_2 = \frac{D^2}{2S^2} = \frac{1}{2}\left[\int_{-\infty}^{+\infty} t^2\frac{p'^2}{p}\,dt - 1\right].$$

For the normal law,

$$-\frac{p'}{p} = \frac{t}{\sigma^2}, \qquad \frac{1}{\lambda_1} = \frac{1}{\sigma^2}, \qquad \frac{D}{\lambda_2} = 2.$$

$$f_1(t) = 1, \qquad f_2(t) = Dt/\sigma^2, \qquad A_1 = 1, \qquad A_2 = 1.$$

Thus for the normal law the most accurate norm is the equal-weighted average and the most accurate deviation that obtained by multiplying each measure by the algebraic theoretical deviation corresponding to its order. As we pointed out before the accuracy of the latter is the same as that of the standard deviation itself.

For the symmetric Pearson law,

$$-\frac{p'}{p} = \frac{2nt}{a^2 \pm t^2},$$

$$f_1(t) = 2\lambda_1 n\frac{a^2 \mp t^2}{(a^2 \pm t^2)^2}, \qquad f_2(t) = \frac{4\lambda_2 n a^2 t}{(a^2 \pm t^2)^2}.$$

If the distribution is supernormal, that is if the number of extreme cases is more than normal the weights for the norm and the weights $\div t$ for the deviation should diminish outwards and for the norm should even become negative for large values of t.

On the other hand, if the distribution is subnormal the weights should increase and become infinite at the boundaries $t = \pm a$. In these cases the weighting to be applied is much too complicated to be of any practical value, aside from the impossibility of knowing beforehand the proper values of a and n. However the most important cases are supernormal rather than the reverse and instead of letting the weights diminish according to a complex law we may take equal weights, for the norm, up to a certain point and then discard all measures outside these limits. Such a norm we shall call a discard-average and in practise a certain outer fraction of the measures is discarded. For the deviation we may discard not the outer but the inner fraction. Our next paragraph deals with such special types.

5. *Special Types of Average.—Discard Average.* Let k be the central fraction of the group retained and let t_1 be the solution of

$$2\int_0^{t_1} p(t)dt = k. \tag{13}$$

Then

$$\begin{aligned} f(t) &= \frac{1}{k}, && -t_1 < t < +t_1 \\ &= 0, && \text{otherwise.} \\ \varphi(t) &= \frac{1}{k}t, && 0 < t < t_1 \\ &= \frac{1}{k}t_1, && t > t_1. \end{aligned}$$

By formula (9)

$$S^2 = \frac{1}{k^2}\left[2\int_0^{t_1} t^2p(t)dt + t_1^2 2\int_{t_1}^{\infty} p(t)dt\right].$$

Let α denote the ratio $2\int_0^{t_1} t^2p(t)dt : kt_1^2$,

$$2\int_0^{t_1} t^2p(t)dt = \alpha kt_1^2, \tag{14}$$

$$S^2 = \frac{t_1^2}{k^2}[1-(1-\alpha)k]. \tag{15}$$

In the case of the median k and t_1 approach 0 together, and

$$S^2 = \frac{1}{4p_0^2}. \tag{15a}$$

In the following discussion for purposes of comparison we shall use the normal Gaussian law and the two extreme Pearson symmetric forms,

$$\text{supernormal,} \qquad p(t) = \frac{2}{\pi}(1 + t^2)^{-2}, \tag{a}$$

$$\text{normal,} \qquad p(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}, \tag{b}$$

$$\text{subnormal,} \qquad p(t) = \frac{3}{4}(1 - t^2), \qquad (t^2 \leqq 1). \tag{c}$$

The accuracy of the median will be, in percentages,

(*a*) 162, (*b*) 63.7, (*c*) 45.

For the quartile-discard average, $k = 1/2$ and the fraction α takes the values,

(*a*) .306, (*b*) .314, (*c*) .323.

Formula (15) becomes

$$S^2 = 2t_1^2(1 + \alpha).$$

The accuracy will be, in percentages,

(*a*) 200,* (*b*) 83.7, (*c*) 63.*

In a supernormal Pearson distribution,

$$p(t) = c(1 + t^2)^{-n}.$$

For the quartile-discard average, when n is large,

$$S^2 = \frac{1.195}{2n}\left[1 + \frac{.851}{n} + \frac{.700}{n^2}\right] = \frac{1.195}{2n - 1.70}.$$

Hence the accuracy of the quartile-discard average will be

$$A = 83.7\frac{2n - 1.70}{2n - 3.00}\text{ per cent.} = 83.7 + \frac{109}{2n - 3}\text{ per cent.}$$

The formula can hardly be used with accuracy when n is as small as 2, but even then it would give the value

$$A = 193.$$

* These values are only rough approximations.

The quartile-discard average will be more accurate than the ordinary mean if

$$2n < 9.3.$$

If σ^2 is average t^2 and q^4 average t^4 then the above condition may be translated into

$$q^4 > 4 \cdot 4\sigma^4,$$

instead of $3\sigma^4$ for a normal law.

Median-Quartile Average.—This average is the mean of the median and the two quartiles,

$$\bar{t} = \frac{1}{3}(Q_1 + M + Q_3), \qquad S^2 = \frac{1}{36}\left(\frac{1}{p_0^2} + \frac{2}{p_0 p_1} + \frac{2}{p_1^2}\right),$$

where $p_0 = p(t_0)$, $p_1 = p(t_1)$ and t_1 is the theoretical quartile deviation.

For a normal law the accuracy is

$$A = 86.0 \text{ per cent.}$$

It appears to be a little more accurate than the quartile-discard average, but we have assumed that the number of observations is large. When the number is small it will be difficult to determine the quartiles exactly, so that, taking everything into consideration, we may say the median-quartile average and the quartile-discard average are about equally accurate.

The most serious objection to the use of any special type of average is that discontinuity is introduced; that is, if the measures are considered as sufficiently normal, none will be discarded; if not, some may be discarded and there will be a finite change in the average. To obviate this difficulty we might use a combination of quartile-discard and ordinary average.

Let p be the weight assigned to the ordinary average.

$q = 1 - p =$ weight assigned to the quartile-discard.

$\sigma^2 =$ mean square deviation.

$N =$ mean numerical deviation.

$P =$ quartile deviation, or probable error.

Then

$$S^2 = p^2\sigma^2 + 2pqP(2N - \tfrac{1}{6}D) + q^2 \tfrac{8}{3} P^2, \tag{16}$$

approximately, assuming average t^2 from 0 to P is $\frac{1}{3}P^2$, average t from 0 to P is $\frac{1}{2}P$. Then we may choose p, q so as to make S^2 minimum.

6. *Special Types of Dispersion Measure. Numerical Deviation.*—In this

case

$$f(t) = +1, \quad t > 0,$$
$$= -1, \quad t < 0,$$

$$\varphi(t) = |t| - N,$$

where N is the theoretical mean numerical deviation. Then, by (9),

$$S^2 = \sigma^2 - N^2.$$

By the definition, succeeding formula 11, the accuracy will be

$$\frac{N^2}{2S^2} = \frac{N^2}{2(\sigma^2 - N^2)} \, .$$

In the case of our three Pearson types (*a*) supernormal, (*b*) normal, (*c*) subnormal, the accuracy, in percentages, will be

(*a*) 34, (*b*) 87.6, (*c*) 118.

Discard Deviation.—In this case we discard the inner portion and then use the mean numerical deviation of the remainder. Under a normal law, if the portion between the quartiles, that is the central half, is discarded the accuracy is 96.3 per cent. Hence this is practically as accurate as the standard deviation and may, in some cases, be more rapidly found as it is a numerical mean and the calculations are made for half the measures only.

Quartile Deviation, etc.—If t is the theoretical deviation the accuracy will be

$$8[tp(t)]^2.$$

For the quartile deviation this is 36.7 per cent.

It will be a maximum when $tp(t)$ is maximum and for a normal law this is given by $t = \sigma$. The values $t = \pm\,\sigma$ practically divide the whole range of measures into two thirds within and one third without.

We may call this the sextile deviation, remembering that it is the outermost sextiles only which are used. Then the accuracy becomes 46.8 per cent. Furthermore it is much easier to find the corresponding standard deviation σ in this case, for theoretically with a normal law

$$\sigma = \left(1 + \frac{1}{30}\right) \text{outer-sextile deviation.}$$

We might also call this the semi-probable error, that is, such that the chances of exceeding it are just one half the chances of not exceeding.

A table is added of the accuracies of the various types when the law is assumed to be normal.

Norms:	
Median	63.7 per cent.
Quartile average $(Q_1 + M + Q_3)/_3$	86.0 " "
Quartile discard average (outer quartiles discarded)	83.7 " "
Mean	100 " "
Dispersion:	
Quartile deviation	36.7 " "
Outer sextile deviation	46.8 " "
Numerical deviation	87.6 " "
Discard deviation (inner quartiles discarded)	96.3 " "
Standard deviation	100 " "

RICE INSTITUTE,
HOUSTON, TEXAS.

SOME DETERMINANT EXPANSIONS.*

By Lepine Hall Rice.

§ 1. There is a recent paper by Sir Thomas Muir† which presents an important general theorem upon the expansion of a determinant. Muir states the theorem summarily as follows:

A determinant can be expressed in terms of minors drawn from four mutually exclusive arrays, two of which are coaxial and complementary to one another.

The discussion leading up to this statement involves bordered determinants. But without reference to such determinants, by following a line of thought suggested by matter contained in §§ 11 and 12 of Muir's paper, it will be found possible not only to prove the theorem in a very simple manner but also to obtain several progressively broader results.

We need first, however, to state the theorem more specifically and in such a manner as to prepare for its extension. With respect to the four arrays it is to be noted that they may be marked out by drawing two lines, one horizontal and the other vertical, across the matrix of the determinant Δ, intersecting on the main diagonal line between two elements thereof; an arrangement which we shall denote as follows:

$$\|\Delta\| \equiv \begin{Vmatrix} A & B \\ C & D \end{Vmatrix};$$

$$A \equiv \begin{Vmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdot & \cdot \\ a_{p1} & \cdots & a_{pp} \end{Vmatrix}, \qquad B \equiv \begin{Vmatrix} b_{11} & \cdots & b_{1q} \\ \cdot & \cdot & \cdot \\ b_{p1} & \cdots & b_{pq} \end{Vmatrix},$$

$$C \equiv \begin{Vmatrix} c_{11} & \cdots & c_{1p} \\ \cdot & \cdot & \cdot \\ c_{q1} & \cdots & c_{qp} \end{Vmatrix}, \qquad D \equiv \begin{Vmatrix} d_{11} & \cdots & d_{1q} \\ \cdot & \cdot & \cdot \\ d_{q1} & \cdots & d_{qq} \end{Vmatrix}, \qquad p + q = n.$$

Further we must particularize with respect to the words "expressed in terms of minors drawn from" the arrays. It is well known that if a set of minors of a determinant Δ, such that their row numbers together are the row numbers of Δ and their column numbers the column numbers of Δ, be taken as the factors of a product to which is prefixed the sign of that term of Δ whose elements are the elements of the main diagonal terms of the minors, this product is identical with the sum of a certain number

* Presented to the American Mathematical Society, Sept. 2, 1919.

† Note on the representation of the expansion of a bordered determinant, by Sir Thomas Muir, LL.D., *Mess. Math.*, No. 566, Vol. xlviii., June, 1918.

of terms of Δ. As elsewhere,* we shall call any two minors of a determinant, which are susceptible of entering into such a set, *conjunctive* minors, and the whole a set of *perjunctive* minors or a *perjunct*; it is a *signed* perjunct if the specified sign is prefixed. When all the minors are of the first order or simply elements of Δ, we have a *transversal* of Δ; if signed, a *term*. The meaning of the phrase in the theorem then is that every possible signed perjunct is to be formed whose minors are four in number and lie one in each of the four arrays. It is understood that any one or more of these minors may be of order zero, with the value 1. And throughout this paper a perjunct will be understood to admit minors of order zero. Let a minor lying wholly in A be called an a-minor, and so for B, C, and D.

We are now ready to restate Muir's theorem as

THEOREM *A*. *If the matrix of any determinant Δ be partitioned, by a horizontal and a vertical line intersecting on the main diagonal line, into four arrays, A, B, C, and D, then Δ can be expanded as the sum of all signed perjuncts composed of an a-minor, a b-minor, a c-minor, and a d-minor.*

§ 2. In the proof of this theorem which we shall now give, and in further proofs in this paper, our line of thought concerns the individual terms of the determinant to be expanded, in their relation to the specified arrays into which the matrix of the determinant is divided.

Consider then any term of Δ. Separate its elements into those lying in A, those lying in B, those lying in C, and those lying in D. The four groups of elements determine by their row and column numbers four minors lying respectively in the four arrays and forming a perjunct which is evidently the only perjunct of four minors lying in the four arrays which contains this term.

Thus the sum of perjuncts specified in the theorem contains nothing but terms of Δ and contains every term once and only once. It is therefore an expansion of Δ.

§ 3. We are immediately led to give the theorem additional breadth by removing the condition that the horizontal and vertical lines must intersect on the main diagonal line, for the proof does not hang upon that condition; and we have

THEOREM 1. *If the matrix of any determinant Δ be partitioned by a horizontal and a vertical line into four arrays, A, B, C, and D, then Δ can be expanded as the sum of all signed perjuncts composed of an a-minor, a b-minor, a c-minor, and a d-minor.*

§ 4. To illustrate this expansion let us take a determinant of order 7, partitioned thus:

$$\Delta \equiv \begin{vmatrix} \|a_{35}\| & \|b_{32}\| \\ \|c_{45}\| & \|d_{42}\| \end{vmatrix}.$$

* P-way determinants, with an application to transvectants, AMERICAN JOURNAL OF MATHEMATICS, Vol. XL, No. 3, July, 1918, p. 242. Cited herein as P-way dets.

(i) Take an a-minor of order 3, a d-minor of order 2, and the b-minor and c-minor determined thereby, the b-minor being of course of order zero while the c-minor is of order 2; example, $\underset{123}{a_{123}}\underset{45}{c_{12}}\underset{12}{d_{34}}$. (ii) Take an a-minor of order 2, a d-minor of order 1 (an element), and the b-element and c-minor of order 3 determined thereby; example, $-\underset{12}{a_{12}}\underset{1}{b_3}\underset{345}{c_{123}}\underset{2}{d_4}$. (iii) Finally, take an a-element (the d-minor now being of order zero), and the b-minor of order 2 and c-minor of order 4 determined by the a-element; example, $\underset{1}{a_1}\underset{12}{b_{23}}\underset{2345}{c_{1234}}$. As a check, we may count up in the result the terms of Δ: $\binom{5}{3}\binom{4}{2}3!\,2!\,2! + \binom{5}{2}\binom{3}{2}\binom{4}{1}\binom{2}{1}2!\,3! + \binom{5}{1}\binom{3}{1}2!\,4! = 7!$.

This procedure is applicable generally. We start by forming all possible perjuncts consisting of one of the largest a-minors and one of the largest d-minors, together with the b-minor and c-minor determined thereby; next we form all possible perjuncts with the a-minor and d-minor one less in order, and the b-minor and c-minor one greater; and so we continue, until the a-minor or the d-minor becomes of order zero.

§ 5. The next generalization consists in removing altogether the restrictions on the manner of partitioning the matrix of Δ into rectangular arrays. Let us call a rectangular array which is a part of the matrix of Δ a *panel* of Δ. Panels may be of any number and each may be of any dimensions so long as all fit together into the square matrix. With slight and obvious changes the former proof covers this more inclusive case, and we have

THEOREM 2. *If the matrix of a determinant Δ be partitioned into panels in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each panel.*

§ 6. A theorem was announced by Albeggiani in 1875 in a paper entitled *Sviluppo di un determinante ad elementi polinomi,** which interests us here for three reasons. First, it can be proved in the manner of § 2 with directness and brevity. Secondly, it can be utilized to establish Theorem 2. And thirdly, it can be generalized from two dimensions to three or more.

As Albeggiani himself pointed out, this theorem applies to any determinant whatever, for polynomial elements can be made out of monomial elements *ad libitum*, either by breaking up the monomial elements or by annexing zero terms. Consider then the general determinant $\Delta = |a_{1n}|$, and put

$$a_{ij} = h_{ij}^{(1)} + h_{ij}^{(2)} + \cdots + h_{ij}^{(r)}.$$

Set up the r determinants

$$\Delta^{(k)} \equiv \begin{vmatrix} h_{11}^{(k)} & \cdots & h_{1n}^{(k)} \\ \cdot & \cdot & \cdot \\ h_{n1}^{(k)} & \cdots & h_{nn}^{(k)} \end{vmatrix}, \qquad k = 1, 2, \cdots, r.$$

* *Giorn. di Batt.*, Vol. 13, p. 1.

Form what we may call a *mixed perjunct* by taking one minor from $\Delta^{(1)}$, a second minor, conjunctive in position, from $\Delta^{(2)}$, and so on to $\Delta^{(r)}$, and prefix the sign determined precisely as it would be determined if all the minors came from one determinant. Then we may state Albeggiani's theorem as follows:

If Δ be any determinant, the sum of all the signed mixed perjuncts from r determinants so formed that the sum of their matrices is the matrix of Δ, is an expansion of Δ.

To prove the theorem, consider any term of Δ. This a-term yields r^n monomials each the product of n h's, which may be called h-terms of Δ. Now obviously we can think of expanding Δ directly into its h-terms without first forming the a-terms. And from that point of view it is clear that any given h-term is to be found in one and only one mixed perjunct. For, separate the elements of this h-term into the $h^{(1)}$'s, the $h^{(2)}$'s, and so on. These groups determine just one mixed perjunct containing this term; and therefore, as the perjuncts contain nothing but h-terms of Δ, we have an expansion of Δ.

§ 7. In order to prove Theorem 2 by means of Albeggiani's theorem, we form $\Delta^{(1)}, \Delta^{(2)}, \cdots, \Delta^{(r)}$ by writing into r blank matrices the r panels of Δ, each in its proper place, and then filling up each matrix with zeros. All minors of $\Delta^{(1)}$ vanish except those lying in the first panel, all minors of $\Delta^{(2)}$ except those lying in the second panel, and so on. The mixed perjuncts which survive are identical with the perjuncts of Δ specified in Theorem 2.

§ 8. Let us next extend Albeggiani's theorem to cubic or 3-way determinants, preparatory to an extension to p-way determinants. Let*

$$\Delta = |a_{\check{\alpha}\hat{\beta}\hat{\gamma}}|_n; \qquad a_{\alpha\beta\gamma} = h^{(1)}_{\alpha\beta\gamma} + h^{(2)}_{\alpha\beta\gamma} + \cdots + h^{(r)}_{\alpha\beta\gamma}.$$

Set up the r determinants

$$\Delta^{(k)} = |h^{(k)}_{\check{\alpha}\hat{\beta}\hat{\gamma}}|_n; \qquad k = 1, 2, \cdots, r.$$

Then we have

THEOREM 3. *If Δ be any 3-way determinant, the sum of all the signed mixed perjuncts from r determinants of the same signancy as Δ, the sum of whose matrices is the matrix of Δ, is an expansion of Δ.*

The proof follows that of § 6 very closely, the introduction of the nonsignant third index giving no trouble.

Defining a *block* as a 3-way rectangular matrix forming a part of the matrix of Δ, we have the

COROLLARY. *If the matrix of a 3-way determinant Δ be partitioned into blocks in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each block.*

* For the notation, see *P*-way dets., §§ 5, 6.

In particular, the blocks may be formed by three mutually perpendicular planes passed through the matrix. The types of perjuncts become here much more numerous than in the case of a 2-way determinant under Theorem 1. In any speical determinant there may be blocks the character of whose elements will simplify the application of the Corollary.

§ 9. Finally, consider the general p-way determinant

$$\Delta = |a_{\alpha\beta\cdots\kappa\lambda}|^{(p)}_{n},$$

in which any or all of the indices may be signant or nonsignant. Put

$$a_{\alpha\beta\cdots\kappa\lambda} = h^{(1)}_{\alpha\beta\cdots\kappa\lambda} + h^{(2)}_{\alpha\beta\cdots\kappa\lambda} + \cdots + h^{(r)}_{\alpha\beta\cdots\kappa\lambda},$$

and form r determinants of the same signancy as Δ:

$$\Delta^{(k)} = |h^{(k)}_{\alpha\beta\cdots\kappa\lambda}|^{(p)}_{n}, \qquad k = 1, 2, \cdots, r.$$

THEOREM 4. *If Δ be any p-way determinant, the sum of all the signed mixed perjuncts from r determinants of the same signancy as Δ, the sum of whose matrices is the matrix of Δ, is an expansion of Δ.*

Proof. First, to show that a signed perjunct consists of a certain number of terms of Δ. That the perjunct consists of transversals of Δ, is clear. It is now to the correspondence of signs that we must look. And it will be perceived that this point is really settled by the known correspondence in the case of a 2-way determinant. For, the argument in that case considers, first, row numbers, next, column numbers, treating both sets in the same way and combining the results. Here we have simply to apply the same argument to each signant index in turn, and to combine the results by taking the product of the signs of the signant ranges.

Secondly, to find any given h-term of Δ in one and only one mixed perjunct. We group the $h^{(1)}$'s, the $h^{(2)}$'s, and so on. Previous reasoning is here followed and the result readily comes, completing the proof.

Extend the definition of a *block* to p dimensions: it is to consist of all those elements for which α has a value found among a fixed set of values $\alpha_1, \alpha_2, \cdots, \alpha_{b_1}$; β, a value found among a set of values $\beta_1, \beta_2, \cdots, \beta_{b_2}$; and so on. The locant of the block is thus

$$\left\{\begin{matrix} \alpha_1\alpha_2 \cdots \alpha_{b_1} \\ \beta_1\beta_2 \cdots \beta_{b_2} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_1\lambda_2 \cdots \lambda_{b_p} \end{matrix}\right\}.$$

We shall then evidently have, under Theorem 4, the

COROLLARY. *If the matrix of a p-way determinant Δ be partitioned into blocks in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each block.*

§ 10. It is important to note that all of the foregoing results apply to permanents as well as to determinants, since the reasoning in no case depends—as does, for instance, the reasoning which establishes the multiplication theorem—upon the vanishing of certain aggregates of terms.

DEPARTMENT OF MATHEMATICS,
MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASS.

A GENERAL IMPLICIT FUNCTION THEOREM WITH AN APPLICATION TO PROBLEMS OF RELATIVE MINIMA.

By K. W. Lamson.

Goursat has given a proof of the existence of a system of solutions of the equations

$$y_i = F_i(y_1, \cdots, y_n;\ z_1, \cdots, z_m) \qquad (i = 1, 2, \cdots, n), \tag{1}$$

where the functions F_i reduce to $y_i^{(0)}$ for $y = y^{(0)}$, $z = z^{(0)}$, and their difference from $y_i^{(0)}$ is of an order higher than the first in the variables y. He has further shown how, under certain conditions, the following system

$$G_i(y_1, \cdots, y_n;\ z_1, \cdots, z_m) = 0 \qquad (i = 1, 2, \cdots, n), \tag{2}$$

can be reduced to the form (1). A system of equations of type (2) arises in the theory of relative extrema of functions of a finite number of variables (referred to as theory I).

Equations (1) and (2) suggest the following problem of implicit functions in the theory of Functions of Lines. Let x, ξ be variables on the continuous range ab, and consider a functional operation $F[y(x), z(x);\ \xi]$ such that to a pair of functions $y(x)$, $z(x)$ and number ξ on ab corresponds a unique real number. Further suppose that $F[y(x), z(x);\ \xi]$ reduces to y_0 when $y = y_0$, $z = z_0$, and that its difference from y_0 is of an order higher than the first, with a suitable definition of order of difference. The subscript i, thought of as a variable with the discrete range $1, 2, \cdots n$, or $1, 2, \cdots m$, has been replaced by the variable ξ with the continuous range ab. The functions $y(x)$, $z(x)$ take the place of the sets of numbers y_i, z_i. To equations (1) and (2) correspond

$$y(\xi) = F[y(x), z(x);\ \xi], \tag{3}$$

$$G[y(x), z(x);\ \xi] = 0. \tag{4}$$

Fréchet uses the term "fonctionelle" for F or G, when ξ is fixed, and the term "functional" has come into use as the English equivalent. For equation (3), Volterra* has suggested an existence proof analogous to that of Goursat for equation (1). An instance of equation (4) occurs in the Calculus of Variations in the case of problems in the plane (referred to as theory II).

The first purpose of this paper is to give an existence proof for equations

* *Leçons sur les Fonctions des Lignes*, p. 71.

which include as special cases equations (1), (2), (3) and (4). Equations (3) and (4), although suggested by (1) and (2) are not generalizations of them in the sense of including them as special cases. The general theory is to include also the systems of equations of type (4) appearing in the space problems of the Calculus of Variations (referred to as theory III). The existence theorems used in the theories I, II and III have similarities in hypothesis, proof and conclusion. In I a solution consists of a set of numbers y_i, a function of the variable i, with the range $i = 1, 2, \cdots n$; in II the solution is a function $y(x)$ of the continuous variable x, with the range $a \leqq x \leqq b$, and in III it is a function $y_i(x)$ of the composite variable (i, x) with the composite ranges $i = 1, 2, \cdots n, a \leqq x \leqq b$. The difference in the three theories lies in the difference in the range of the independent variable. Any general theory which includes the three as special cases will introduce a range which will specialize to the three just mentioned. For two reasons it has seemed best not to attempt to abstract common properties from these ranges, but to introduce the general range* of E. H. MOORE, not defined and on which no postulates bear explicitly. In the first place the dissimilarities make it hard to find useful common properties, and in the second place, the general theory is not to exclude problems involving double integrals or combinations of integrals and sums. The general range is a set $\mathfrak{P}$ of elements p, and the functions to be considered are such that to each p corresponds a real number $y(p)$.

Replace the i of equations (1) and (2) and the x of (3) and (4) by p. This leads to the equations

$$y(p) = F[y(q), z(q); p], \tag{5}$$

$$G[y(q), z(q); p] = 0, \tag{6}$$

where q has the range $\mathfrak{P}$ and where, by means of F and G, to each p and pair of functions y and z in a certain class $\mathfrak{M}$ of functions there corresponds a unique real number.

In § 1 below the basis and postulates for the solution of equations (5) and a special form of (6) are set down. In §§ 2, 3 are lemmas leading to the solution of (5) and to the reduction of (6) to the form (5). The last section of the paper contains an application to the problem of Lagrange in the Calculus of Variations.

§ 1. *The Basis.*

The independent variable of the theory has the general range $\mathfrak{P}$. An element of $\mathfrak{P}$ will be denoted by one of the letters p, q. The functions entering the theory belong to a class $\mathfrak{M}$, whose elements are real single-valued functions $y(p)$ or $z(p)$. In theory I the class $\mathfrak{M}$ is the set of n-

* Bolza, *Bulletin of the American Mathematical Society*, Vol. 16 (1910), p. 403; also *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. 23 (1914), p. 251.

partite numbers or of points in n-space. In theories II and III $\mathfrak{M}$ is the class of functions or curves in the plane and in $(n+1)$-space respectively, continuous with their first derivatives. To each element y of $\mathfrak{M}$ corresponds a positive or zero number, the "modulus" of y, which will be denoted by $\| y \|$. In theory I the modulus is interpreted as the largest of the numbers y_i, or as the distance of the point $(y_1, \cdots, y_n)$ from the origin. In theories II and III the modulus is interpreted as the number defining a neighborhood of the first order, namely the maximum absolute value of the functional value and of the derivative. In the general theory the modulus is not defined and is subject to postulates. These postulates and those on $\mathfrak{M}$ will be shown in § 4 to be satisfied in the case of the Lagrange problem.

Postulate 1. $\mathfrak{M}$ is linear, that is, contains all functions of the form $c_1 y_1 + c_2 y_2$, where c_1 and c_2 are real numbers, provided y_1 and y_2 are themselves in $\mathfrak{M}$.

Postulate 2. $\| y_1 + y_2 \| \leqq \| y_1 \| + \| y_2 \|$.

Postulate 3. $\| cy \| = |c| \, \| y \|$, for every real number c.

Postulate 4. If $\| y \| = 0$, then $y(p) = 0$ for every p.

THEOREM 1. *If $\{y_i\}$ and $\{y'_i\}$ are sequences, and y and y' are functions, such that $\lim_{i=\infty} \| y - y_i \| = \lim_{i=\infty} \| y_i - y'_i \| = 0$; and if $\lim_{i-\infty} \| y_i - y'_i \| < b$, then $\| y - y' \| \leqq b$.*

This theorem follows at once from the preceding postulates.

Definition. The sequence $\{y_i\}$ is defined to be a Cauchy sequence if

$$\lim_{i=\infty,\, j=\infty} \| y_i - y_j \| = 0.$$

The sequence $\{y_i\}$ is said to have a limit y if $\lim_{i=\infty} \| y - y_i \| = 0$. The uniqueness of this limit is a result of postulates 2, 3, 4.

Postulate 5. For every Cauchy sequence in $\mathfrak{M}$ there exists a function in $\mathfrak{M}$ which is the limit of this sequence.

Definition. The symbol $(\bar{y})_a$ denotes the totality of functions y of $\mathfrak{M}$ such that $\| y - \bar{y} \| < a$.

Consider $F[y_1, \cdots, y_\kappa; p]$ real and single-valued for y_i in $(\bar{y}_i)_a$ $(i = 1, \cdots \kappa)$ and p in $\mathfrak{P}$, and such that when $y_1, \cdots, y_\kappa$ are fixed the resulting function of p is in $\mathfrak{M}$.

Definition. The functional F is continuous at a set of arguments $(y'_1, \cdots, y'_\kappa)$ if for every ϵ there exists a δ such that

$$\| F[y_1, \cdots, y_\kappa; p] - F[y'_1, \cdots, y'_\kappa; p] \| < \epsilon$$

whenever y_i is in $(y'_i)_\delta$.

§ 2. *Solution of the equation* $y(p) = F[y, z; p]$.

The proof of the existence of a solution, $y(p)$, of the equation

$$y(p) = F[y, z; p] \tag{5}$$

is similar to that given by Goursat,* who used the method of successive approximations to treat equation (1). Let y_0 and z_0 be two functions of the class $\mathfrak{M}$. The functional $F[y, z; p]$ is supposed to be real and single-valued for all elements $(y, z; p)$ for which y is in a neighborhood $(y_0)_a$, z in $(z_0)_a$, and p in $\mathfrak{P}$, and to have the property that when y and z are fixed in its range of definition the resulting function of p is also in $\mathfrak{M}$. It has further the properties

(1) $F[y_0, z_0; p] = y_0(p)$ for every p in $\mathfrak{P}$;

(2) it is continuous in y and z at each element y', z', in its range of definition;
(3) there exists a constant $0 < K < 1$ such that

$$\|F[y_1, z; p] - F[y_2, z; p]\| < K\|y_1 - y_2\|$$

whenever (y_1, z) and (y_2, z) are in the range for which F is defined. This condition will be referred to as the Lipschitz condition.

Define a sequence of successive approximations by the equations

$$y_1 = F[y_0, z; p] \tag{7}$$

$$y_{i+1} = F[y_i, z; p] \qquad (i = 1, 2, 3, \cdots), \tag{8}$$

which is possible whenever every y_i is in $(y_0)_a$. It will first be shown that a neighborhood $(z_0)_{a_1}$ with $a_1 \leqq a$ can be chosen so that the elements of the sequence are well defined whenever z is in $(z_0)_{a_1}$.

LEMMA 1. *There exists a positive constant $a_1 \leqq a$ such that for z in $(z_0)_{a_1}$, and for every i, y_i is in $(y_0)_a$.*

To prove this, use the continuity of F in z, and choose $a_1 \leqq a$, so that

$$\|y_1 - y_0\| = \|F[y_0, z; p] - F[y_0, z_0; p]\| < a(1 - K).$$

From the Lipschitz condition, if y is in $(y_0)_a$,

$$\|F[y, z; p] - F[y_0, z; p]\| < K\|y - y_0\|.$$

From the addition of $\|F[y_0, z; p] - y_0\|$ to both sides, and from Postulate 2, follows

$$\|F[y, z; p] - y_0\| < K\|y - y_0\| + a(1 - K). \tag{9}$$

In particular, putting $y = y_1$, this becomes

$$\|y_2 - y_0\| < K\|y_1 - y_0\| + a(1 - K) < a.$$

To complete the induction proof, assume $\|y_i - y_0\| < a$, and put y_i in (9).

* Goursat, *Bulletin de la Société Mathématique de France*, Vol. 31 (1903), p. 184. Bliss, *Princeton Colloquium Lectures*, p. 8.

LEMMA 2. *The sequence* $\{y_i\}$ *is a Cauchy sequence and its limit,* y, *(Postulate 5) is in* $(y_0)_a$.

To prove this, the convergence of the series $\sum_i \|y_{i+1} - y_i\|$ is first shown, by using $\sum_i K^i a$ as a dominating series. From the definition of y_2 and y_1, and from the Lipschitz condition,

$$\|y_2 - y_1\| = \|F[y_1, z;\ p] - F[y_0, z;\ p]\| < K\|y_1 - y_0\| < Ka.$$

To complete the induction proof, assume

$$\|y_{i+1} - y_i\| < K^i a,$$

and apply the Lipschitz condition to $\|y_{i+2} - y_{i+1}\|$.

The convergence of $\Sigma\|y_{i+1} - y_i\|$, and Postulate 2 imply that the sequence $\{y_i\}$ is a Cauchy sequence. From Theorem 1 it follows that the limit y of $\{y_i\}$ is in $(y_0)_a$.

LEMMA 3. *The equation* (5) *is satisfied by the limit* y *of Lemma* 2.

For from the definition of y_i, and from Lemma 2,

$$\lim_{i=\infty} \|y - y_i\| = \lim_{i=\infty} \|y - F[y_i, z;\ p]\| = 0. \tag{10}$$

From the continuity of F,

$$\lim_{i=\infty} \|F[y_i, z;\ p] - F[y, z;\ p]\| = 0, \tag{10a}$$

and from the addition of (10) and (10a), and the application of Postulates 2 and 4,

$$y = F[y, z;\ p].$$

LEMMA 4. *The solution* y *of equation* (5) *described in the preceding lemmas is the only one in* $(y_0)_a$ *corresponding to a* z *in* $(z_0)_{a_1}$.

For the proof, assume two solutions, and apply the Lipschitz condition to their difference, using Postulate 4.

LEMMA 5. *As a functional of* z, y *is continuous in the neighborhood* $(z_0)_{a_1}$.

It is necessary to show that if $\|z - z'\|$ is small, then $\|y - y'\|$ is small, where y and y' are the solutions corresponding to z and z' respectively.

From Postulate 2,

$$\begin{aligned}\|y - y'\| &= \|F[y, z;\ p] - F[y', z';\ p]\| \\ &\leqq \|F[y, z; p] - F[y', z; p]\| + \|F[y', z; p] - F[y', z'; p]\| \\ &\leqq K\|y - y'\| + \|F[y', z;\ p] - F[y', z';\ p]\|.\end{aligned}$$

From the continuity in z, the last term can be made less than an ϵ as required, whence

$$\|y - y'\| < \frac{\epsilon}{1-K}.$$

The results of this section may be summed up in the following

THEOREM 2. *When $F[y, z; p]$ has the solution $(y_0, z_0; p)$ and the properties described at the beginning of this section for elements $(y, z; p)$ with y in $(y_0)_a$, z in $(z_0)_{a_1}$ and p in $\mathfrak{P}$, there exists a constant $a_1 \leqq a$ such that the equation*

$$y = F[y, z; p]$$

has one and only one solution $y = Y[z; p]$ for each z in the neighborhood $(z_0)_{a_1}$. The functional $Y[z; p]$ so defined has the value $y = y_0$ for $z = z_0$ and is continuous at $z = z_0$.

§ 3. *The equation $G[y; p] = z(p)$.*

In order to transform equation (2) to the form (1), Goursat* assumes first that the derivatives $\partial G_i/\partial y_j$ exist and are continuous, and second that the functional determinant is different from zero for those values of y_i and z_i for which the G_i vanish. The equation (6) will be taken in the less general form,

$$G[y; p] = z(p), \tag{11}$$

which is to be solved for y, given that

$$G[y_0; p] = z_0(p).$$

The equation (11) will be transformed to the form (5) treated in the preceding section, by a procedure following that of Goursat. Before prescribing the properties of the functional G it will be useful to describe those of a functional $A[y_1, y_2, \eta; p]$ which will be called a difference function for reasons which will presently appear. At each element $(y_1, y_2, \eta; p)$ with y_1 and y_2 in $(y_0)_a$, η in $\mathfrak{M}$, and p in $\mathfrak{P}$ the functional A has a single real value, and when the first three of its arguments are fixed defines a function of the class $\mathfrak{M}$. It has furthermore the following properties:

(1) it is linear in η, that is,

$$A[c_1\eta_1 + c_2\eta_2] = c_1A[\eta_1] + c_2A[\eta_2]$$

where η_1 and η_2 are functions of the class $\mathfrak{M}$ and c_1 and c_2 are constants. The three arguments other than η are suppressed for the moment in this equation;

(2) There exists a constant M such that

$$\| A[y_1, y_2, \eta; p] \| \leqq M \| \eta \|$$

whenever $(y_1, y_2, \eta; p)$ is in the set for which A is defined;†

(3) the functional A is uniformly continuous in (y_1, y_2) at (y_0, y_0) with

* Loc. cit., p. 191.

† Riesz, *Annales Scientifique de L'École Normale Supérieure*, 3me Série, Vol. 31 (1914), p. 10.

respect to the set of admissible arguments η for which $||\eta|| = 1$, that is for every given ϵ there exists a δ such that

$$||A[y_1, y_2, \eta; p] - A[y_0, y_0, \eta; p]|| < \epsilon$$

whenever y_1 and y_2 are in $(y_0)_\delta$, η is in $\mathfrak{M}$.

The functional $G[y; p]$ is supposed to be real single valued for all arguments (y, p) such that y is in $(y_0)_a$ and p in $\mathfrak{P}$, and to have the usual property that it is in the class $\mathfrak{M}$ when the argument y is fixed. It has furthermore a difference function A of the kind described above such that

$$G[y_1; p] - G[y_2; p] = A[y_1, y_2, y_1 - y_2; p]$$

whenever (y_1, p) and (y_2, p) are elements in the domain of definition of G. The functional $A[y_0, y_0, \eta; p]$ is called the differential of G at y_0. Since y_0 is a fixed element of the class $\mathfrak{M}$ the differential is a function of η and p alone.

The use which Goursat makes of his hypothesis concerning the non-vanishing of the functional determinant suggests the assumption that A has a "reciprocal" for $y_1 = y_2 = y_0$, namely that there exists a functional $\bar{A}[\eta; p]$ such that

$$\bar{A}[A[y_0, y_0, \eta; q]; p] = \eta(p)$$

$\bar{A}[\eta; p]$ has the properties (1) and (2) prescribed for A, where $\bar{M}$ denotes the number corresponding to the M of property (2). It has the further property that it vanishes identically in p only when $\eta(p) = 0$ for every p.

LEMMA 6. *The functional F defined by the equation*

$$F[y, z; p] = y - \bar{A}[G[y; p] - z; p]$$

has the properties of the functional F of § 2 near the element (y_0, z_0) where

$$z_0 = G[y_0; p].$$

As to the property (1) of § 2, it follows from the definition of F given in this lemma that

$$F[y_0, z_0; p] = y_0 - \bar{A}[0; p] = y_0.$$

The continuity, property 2, is proved by these inequalities,

$$\begin{aligned} ||F[y, z; p] - F[y', z'; p] & \\ &= ||y - y' + \bar{A}[G[y; p] - G[y'; p] - z + z'; p]|| \\ &\leqq ||y - y'|| + \bar{M}||G[y; p] - G[y'; p] - z + z'|| \\ &\leqq (1 + M\bar{M})||y - y'|| + ||z - z'||. \end{aligned}$$

To find the K of property 3, use the linearity of the functional $\bar{A}$.

$$\begin{aligned} ||F[y, z; p] - F[y', z; p]|| &= ||y - y' - \bar{A}[G[y; q] - G[y'; q]; p]|| \\ &= ||y - y' - \bar{A}[A[y, y', y - y'; q] - A[y_0, y_0, y - y'; q] \\ &\qquad + A[y_0, y_0, y - y'; q]; p]||. \end{aligned}$$

From linearity, again, from the fact that $\bar{A}$ is the reciprocal of A, and from Postulate 3, $\|-y\| = \|y\|$, this expression reduces to

$$\|\bar{A}[A[y, y', y-y'; q] - A[y_0, y_0, y-y'; q]; p].$$

Because $\bar{A}$ is bounded, this is less than

$$\bar{M}\left\|A\left[y, y', \frac{y-y'}{\|y-y'\|}; p\right] - A\left[y_0, y_0, \frac{y-y'}{\|y-y'\|}; p\right]\right\| \|y-y'\|.$$

The number a of Lemma 1 is then chosen to make the coefficient of $\|y-y'\|$ less than $K < 1$.

THEOREM 3. *The solution of the equation*

$$y = F[y, z; p] \tag{5}$$

where F is defined in Lemma 6, satisfies uniquely the equation

$$G[y; p] = z(p), \tag{11}$$

and is continuous as a functional of z.

For, from the definition of F, (5) reduces to

$$\bar{A}[G[y; q] - z; p] = 0$$

and since $\bar{A}[\eta; p]$ vanishes identically only when $\eta(p) \equiv 0$, it follows that

$$G[y; p] \equiv z(p).$$

Any other function y', a solution of (11), would make F reduce to y', and would satisfy (5). But the solution of (5) is unique (Lemma 4). The solutions of (5) and (11) have been shown to be the same, and the solution of (5) is continuous (Lemma 5). This proves the continuity asserted in the theorem.

§ 4. *An Application to the Calculus of Variations.*

The theorem of § 3 will now be applied to the differential equations of the problem of Lagrange in the Calculus of Variations. For this problem the functions y in the integral

$$\int_a^b f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx,$$

to be minimized are subject to two sets of conditions. They must satisfy, first, the $m < n$ differential equations,

$$\varphi_\alpha(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \qquad (\alpha = 1, \cdots, m), \tag{12}$$

and second, the end conditions,

$$y_i(a) - h_i = 0, \tag{13}$$

$$y_i(b) - k_i = 0 \qquad (i = 1, \cdots, n). \tag{14}$$

The equation (12) may be regarded as a single equation in the composite variable (α, x), whose range is a subset of the range of elements (i, x) where $i = 1, 2, \cdots n, a \leqq x \leqq b$.

Bliss* has given a treatment of a problem of which this is a special case by adjoining to (12) the $n - m$ new equations

$$\varphi_r(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = Z_r(x) \quad (r = m + 1, \cdots, n). \tag{15}$$

In (15) the functions φ_r are arbitrary except that they are to be chosen so that the determinant $|\partial\varphi_i/\partial y_j'|$ is different from zero at every point of the minimizing arc to be studied. Equations (12) and (15) can then be written together in the single equation

$$\varphi_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = Z_i(x) \quad (i = 1, 2, \cdots, n), \tag{16}$$

with the understanding that $Z_i = 0$ identically in x, for $i \leqq m$.

Consider now a system of solutions $y_i^{(0)}(x)$, $Z_i^{(0)}(x)$ of class C' of the equations (16). In a neighborhood of the elements (x, y, y') of this solution the functions φ_i are supposed to have continuous first and second partial derivatives, and along the solution itself the functional determinant $|\partial\varphi_i/\partial y_j'|$ is different from zero. The partial derivatives $\partial\varphi_i/\partial y_j$ and $\partial\varphi_i/\partial y_j'$ will henceforth be denoted by φ_{ij} and ψ_{ij}, and their values at $x = a$, by $\varphi_{ij}(a)$ and $\psi_{ij}(a)$. It is proposed to show that the problem of determining a system of solutions of the equations (16) with initial conditions (13) is a special case of the theorem proved in § 3.

Equations (13) and (16) together are equivalent to the single system

$$G[y(q);\ p] = z(p),$$

where the independent variables are $p = (i, x)$, $q = (j, x_1)$ and G is the functional in the first member of the equation

$$\sum_j \psi_{ij}(a)(y_j(a) - h_j) + \int_a^x \varphi_i(x_1, y, y')dx_1 = z_i(x) \quad (i = 1, \cdots, n). \tag{17}$$

Equations (13) have been multiplied by a matrix of rank n. The z_i appearing in (17) are the integrals from a to x of the functions $Z_i(x)$ in (16), and so vanish for $x = a$. Equations (14) are discussed later.

The general theory of the preceding sections will be applied to the solution of (17) for y when z is given. With the $y^{(0)}$ which minimizes the integral is associated a $z^{(0)}$ by equations (17), and it is in a first order neighborhood of these functions that a solution is to be found. The range $\mathfrak{P}$ is specified to be the set of elements (i, x), $(i = 1, \cdots, n;\ a \leqq x \leqq b)$. The class $\mathfrak{M}$ is the class of functions $y_i(x)$ which for each i are continuous with their first derivatives on the interval ab. The modulus, $||y||$, is

* *Transactions of the American Mathematical Society*, Vol. 19 (1918), p. 307.

the maximum of the absolute values of y_i and y'_i $(i = 1, \cdots, n)$. The functional $G[y;\, p]$ is the left-hand member of equations (17).

It remains to exhibit the differential A, its reciprocal $\bar{A}$, and to prove that the postulates of § 1 and hypotheses of § 2 are satisfied. Postulates 1–4 are immediately seen to be satisfied. Postulate 5 can be proven from the fact that convergence of the moduli of a sequence of functions of $\mathfrak{M}$ implies the uniform convergence of the functions and of their first derivatives.

The differential A is given for the function (17) by Taylor's formula* in the form

$$\sum_j \psi_{ij}(a)\eta_j(a) + \sum_j \int_a^x \{C_{ij}(x_1)\eta_j(x_1) + C'_{ij}(x_1)\eta'_j(x_1)\}dx_1, \tag{18}$$

where

$$C_{ij}(x_1) = \int_0^1 \varphi_{ij}(x_1,\, y^{(1)} + u(y^{(2)} - y^{(1)}),\, y^{(1)\prime} + u(y^{(2)\prime} - y^{(1)\prime}))du,$$

$$C'_{ij}(x_1) = \int_0^1 \psi_{ij}(x_1,\, y^{(1)} + u(y^{(2)} - y^{(1)}),\, y^{(1)\prime} + u(y^{(2)\prime} - y^{(1)\prime}))du.$$

In C and C', $y^{(1)}$ and $y^{(2)}$ are the arguments of the functional A, and are in a first order neighborhood of the extremal $y^{(0)}$ such that the determinant $|\psi_{ij}| \neq 0$, and φ is defined. When $y^{(1)} = y^{(2)} = y^{(0)}$, A reduces to

$$\sum_j \psi_{ij}(a)\eta_j(a) + \sum_j \int_a^x \{\varphi_{ij}\eta_j(x_1) + \psi_{ij}\eta'_j(x_1)\}dx_1. \tag{19}$$

To exhibit the reciprocal $\bar{A}$ is to define an operation which will reduce (19) to $\eta_k(x)$. This operation will be taken in the form

$$\bar{A} = \sum_i \left[l_{ki}(x)\eta_i(a) + \int_a^x \lambda_{ki}(x,\, x_1)\eta_i(x_1)dx_1 + \nu_{ki}(x)\eta_i(x) \right]$$

with suitably chosen functions l, λ, ν, and it is to be proved that when the functions $\eta(q) = \eta_i(x_1)$ of the variable $q = (i,\, x_1)$ is replaced by A in this expression the result is $\eta(p)$ with $p = (k,\, x)$. To distinguish variables of integration from each other and from limits of integration, the notations x, x_1, x_2 are used. Summations are from 1 to n. To choose the functions l, λ, ν operate as follows. Put $x = a$ in (18) and multiply by undetermined factors $l_{ki}(x)$. Form (18) for x_1, $(a < x_1 < x)$, multiply by $\lambda_{ki}(x,\, x_1)$, and integrate from a to x. For $x_1 = x$, multiply by $\nu_{ki}(x)$. Add the terms so formed and sum as to i.

A method of choosing the functions l, λ and ν is to be given so that the expression,

$$\sum_{ij} \left[\left\{ l_{ki}(x) + \int_a^x \lambda_{ki}(x,\, x_1)dx_1 + \nu_{ki}(x) \right\} \psi_{ij}(a)n_j(a) \right.$$

* Jordan, *Cours d'Analyse*, 2d ed., Vol. 1, p. 247.

$$+\int_a^x \lambda_{ki}(x, x_1)\int_a^{x_1}(\varphi_{ij}\eta_j+\psi_{ij}\eta_j')_{x_2}dx_2dx_1$$
$$+\int_a^x \nu_{ki}(x)\{\varphi_{ij}\eta_j+\psi_{ij}\eta_j'\}_{x_2}dx_2\Bigg],$$

whose formation was described in the preceding paragraph, reduces to $\eta_k(x)$. By the change in order of integration in the second term, and the combination of the last two terms, this becomes

$$(20)\quad \sum_{ij}\Bigg[\Big\{l_{ki}(x)+\int_a^x \lambda_{ki}(x, x_1)dx_1+\nu_{ki}(x)\Big\}\psi_{ij}^{(a)}\eta_j(a)$$
$$+\int_a^x(\varphi_{ij}\eta_j+\psi_{ij}\eta_j')_{x_2}\Big\{\int_{x_2}^x \lambda_{ki}(x, x_1)dx_1+\nu_{ki}(x)\Big\}dx_2\Bigg].$$

A set of auxiliary functions $\mu_k(x, x_2)$ may be defined by means of the equations

$$(21)\qquad \int_{x_2}^x \lambda_{ki}(x, x_1)dx_1+\nu_{ki}(x)=\mu_{ki}(x, x_2)\quad (\kappa, i=1, 2, \cdots, n).$$

From (21) and the integration of the last term by parts, (20) is seen to become

$$(22)\quad \sum_{ij}\Bigg[\Big\{l_{ki}(x)+\int_a^x \lambda_{ki}(x, x_1)dx_1+\nu_{ki}(x)\Big\}\psi_{ij}(a)n_j(a)$$
$$+\int_a^x\Big\{\mu_{ki}(x, x_2)\psi_{ij}(x_2)-\int_a^{x_2}\mu_{ki}(x, x_1)\varphi_{ij}(x_1)dx_1\Big\}\eta_j'dx_2$$
$$+\eta_j(x)\int_a^x \mu_{ki}(x, x_1)\varphi_{ij}(x)dx_1\Bigg].$$

Next it will be shown that the functions $\mu_{ki}(x, x_1)$ can be so chosen that the brace under the integral in the second term is independent of x_2 and therefore equal to a function $\kappa_{ki}(x)$ satisfying the following equation:

$$(23)\qquad \sum_i \mu_{ki}(x, x_2)\psi_{ij}(x_2)=\sum_i\int_a^{x_2}\mu_{ki}(x, x_1)\psi_{ij}(x_1)dx_1+\kappa_{kj}(x),$$
$$(j=1, \cdots, n).$$

The differentiation of (23) for x_2 as it stands would imply the existence of y''. To avoid this replace the μ's by linear combinations of them, $v_{kj}(x, x_2)$, determined by the following equations,

$$(24)\qquad v_{kj}(x, x_2)=\sum_i \mu_{ki}(x, x_2)\psi_{ij}(x_2).$$

The solution of these for the functions μ is possible since $|\psi_{ij}|\neq 0$, and it gives

$$(25)\qquad \mu_{ki}(x, x_2)=\sum_r c_{kr}(x_2)\mathring{v}_{ri}(x, x_2).$$

From (24) and (25), equation (23) becomes

$$v_{kj}(x, x_2) = \sum_{ir} \int_a^{x_2} c_{kr}(x_1) v_{ri}(x, x_1) \varphi_{ij}(x_1) dx_1 + \kappa_{kj}(x).$$

In this equation the right member is differentiable for x_2, and the equations for the determination of the functions v_{kj} may be written in the form

$$(26) \qquad \frac{d}{dx_2} v_{kj}(x, x_2) = \sum_{ir} c_{kr}(x_2) v_{ri}(x, x_2) \varphi_{ij}(x_2).$$

These are linear differential equations which determine $v_{kj}(x, x_2)$ uniquely subject to the initial conditions,

$$(27) \qquad v_{kj}(x, x) = \delta_{kj},$$

where δ_{kj} is unity when $\kappa = j$ and zero otherwise. When the functions v_{kj} are known the μ's are given by (25), the κ's by (23) and the λ's and v's by (21). With the help of (23), (24) and (27) the expression (22) may be replaced by

$$(28) \quad \sum_{ij} \left[\left\{ l_{ki}(x) + \nu_{ki}(x) + \int_a^x \lambda_{ki}(x, x_1) dx_1 \right\} \psi_{ij}(a) \eta_j(a) \right] - \sum_j \kappa_{kj}(x) \eta_j(a) + \eta_k(x).$$

The functions l may now be determined by the equation

$$(29) \quad \sum_i l_{ki}(x) \psi_{ij}(a) = \kappa_{kj}(x) - \sum_i \nu_{ki}(x) \psi_{ij}(a) - \sum_i \psi_{ij}(a) \int_a^x \lambda_{ki}(x, x_1) dx_1$$

so that everything in the expression (28) disappears except $\eta_k(x)$. This result is formulated in the following definition and theorem.

Definition. The differential $A[y_0, y_0, \eta;\, p]$ of the functional $G[y;\, q]$ in the equation (17) for the problem of Lagrange is the expression

$$(30) \qquad \sum_j \psi_{ij}(a) \eta_j(a) + \sum_j \int_a^x (\varphi_{ij}\eta_j + \psi_{ij}\eta_j')_{x_1} dx_1.$$

The functional $\bar{A}[\eta;\, p]$ is given by the formula

$$(31) \qquad \sum_i \left[l_{ki}(x) n_i(a) + \int_a^x \lambda_{ki}(x, x_1) \eta_i(x_1) dx_1 + \nu_{ki}(x) \eta_i(x) \right].$$

In this definition the functions φ_{ij} and ψ_{ij} are formed for the extremal $y^{(0)}$, the functions λ and ν are determined by the equations (26), (27), (25) and (21), and the functions l by (29).

THEOREM 4. *The functional $\bar{A}$ is the reciprocal of A, that is if the η in* (31) *is replaced by the function* (30), *then* (31) *will reduce to* $\eta_k(x)$.

The differential A given by (30) is seen to satisfy the first and second assumptions of § 3. The reciprocal $\bar{A}$ is also seen to satisfy these assump-

tions. The third assumption as to A follows from the continuity properties of φ, and from the mean value theorem. It remains to show that the reciprocal vanishes identically only with the argument η.

LEMMA 7. *If the functions $\eta_i(x)$ are continuous with their first derivatives on the interval ab, and if the equation*

$$(31)\qquad \sum_i \left[l_{ki}(x)\eta_i(a) + \int_a^x \lambda_{ki}(x, x_1)\eta_i(x_1)dx_1 + \nu_{ki}(x)\eta_i(x) \right] = 0$$

holds identically in κ and x, it follows that $\eta_i(x) = 0$ identically in i and x.

To prove this, put $x = a$. From (29) with the help of equations (21) and (23) for $x = x_2 = a$, it follows that $l_{ki}(a) = 0$, and from (24) and (27) it is seen that $|\nu_{ki}(a)| \neq 0$. Therefore $\eta_k(a) = 0$ identically in κ, and it is correct to write

$$\eta_i(x_1) = \int_a^{x_1} \eta_i'(x_2)dx_2.$$

From (31) then

$$\sum_i \int_a^x \lambda_{ki}(x, x_1) \int_a^{x_1} \eta_i'(x_2)dx_2dx_1 + \nu_{ki}(x) \int_a^x \eta_i'(x_2)dx_2 = 0.$$

By change of order of integration, combination of terms and the use of (21), this becomes

$$(32)\qquad \sum_i \int_a^x \mu_{ki}(x, x_1)\eta_i'(x_1)dx_1 = 0.$$

From the theory of differential equations, the solutions of equations (26), and hence also the functions $\mu_k(x, x_1)$, are differentiable for x. Then differentiation of (32) with respect to x gives

$$(33)\qquad \sum_i \mu_{ki}(x, x)\eta_i'(x) = -\sum_i \int_a^x \frac{\partial}{\partial x}\mu_{ki}(x, x_1)\eta_i'(x_1)dx_1.$$

After multiplying by $\bar{\mu}_{rk}(x)$, the matrix reciprocal to $\mu_{ki}(x, x)$, summing with respect to κ and setting

$$-\sum_k \bar{\mu}_{rk}(x) \frac{\partial}{\partial x}\mu_{ki}(x, x_1) = \sigma_{ri}(x, x_1)$$

the equations (33) give

$$(34)\qquad \eta_r'(x) = \sum_i \int_a^x \sigma_{ri}(x, x_1)\eta_i'(x_1)dx_1.$$

The proof that no solution of (32) exists except $\eta_r'(x)$ identically zero is a slight modification of the corresponding proof for Volterra's integral equation.* If M and m are the maxima of $|\sigma_{ri}(x, x_1)|$ and $n_i'(x)$ respectively, for $r, i = 1, 2 \cdots n$ and values of x and x_1 on the interval ab, the

* Bôcher, *An Introduction to the Study of Integral Equations*, p. 15.

equations (34) give

$$m \leqq \int_a^x nMm dx = nMm(x - a),$$

and by repeated applications of this inequality it follows that

$$m \leqq n^\alpha M^\alpha m \frac{(x - a)^\alpha}{\alpha !}$$

for every positive integer α. As this last expression approaches zero with increasing α, it follows that

$$\eta'_r(x) = 0, \qquad a \leqq x \leqq b, \qquad r = 1, \cdots, n.$$

Since $\eta_r(a) = 0$, it is true that $\eta_r(x) \equiv 0$, as stated in the lemma.

The postulates and hypotheses of the general theory have been proved to be satisfied in the case of the Lagrange problem. The results of this section may be stated in the following theorem.

THEOREM. *Under the hypotheses made at the beginning of this section the system of equations*

$$\varphi_i(x, y_1, \cdots, y_n, y'_1, \cdots, y'_n) = Z_i(x) \qquad (i = 1, \cdots, n)$$

with the initial conditions $y_i(a) = h_i$, $(i = 1, \cdots, n)$, *is equivalent to the single equation*

$$\sum_j \psi_{ij}(a)[y_j(a) - h_j] + \int_a^x \varphi_i dx = z_i(x).$$

This has the form

$$G[y(q);\, p] = z(p)$$

where p and q represent the pairs $p = (i, x)$, $q = (j, x)$. If $y^{(0)}(q)$, $z^{(0)}(p)$ is an initial solution of the last equation with properties as prescribed above, then there exist two neighborhoods $(y^{(0)})_a$ and $(z^{(0)})_{a_1}$ such that to every $z(p)$ in the latter there corresponds one and but one solution $y(q)$ in $(y^{(0)})_a$. The functional $y(q) = Y[z;\, q]$ so defined is continuous in $(z^{(0)})_{a_1}$ and reduces to $y = y^{(0)}$ for $z = z^{(0)}$.

THE UNIVERSITY OF CHICAGO.

ON THE LAPLACE-POISSON MIXED EQUATION.

By R. F. Borden.

Introduction.

We designate as the Laplace-Poisson mixed equation, the equation*

$$(1) \qquad f'(x+1) + p(x)f'(x) + q(x)f(x+1) + m(x)f(x) = 0.$$

which was first studied by Poisson†, and which is analogous in form and in theory to the Laplace partial differential equation

$$s + ap + bq + cz = 0.$$

Poisson finds solutions in finite form by means of transformations analogous to those used by Laplace in solving the differential equation written above. These transformations put the mixed equation into equations of the same form, viz:

$$F'(x+1) + P(x)F'(x) + Q(x)F(x+1) + M(x)F(x) = 0.$$

When certain relations exist between the coefficients of one of the transformed equations, Poisson solves that equation by the standard methods of solving linear difference and differential equations of the first order, and then obtains the solution of the original equation by reversing the transformations.

The remark of Poisson, that the theory of this type of equation is but little advanced, still holds true more than a century later. In this paper

* The coefficients $p(x)$, $q(x)$, and $m(x)$ are analytic functions of the real or complex variable x.

† *Jour. de l'École Polytechnique*, t. 6 (1806), pp. 127–141. See also Lacroix, "Traité du Calcul," 3d ed., Vol. 3, pp. 575–600, for the work of Poisson and other early investigators in the field. Other papers on mixed equations are the following: Vernier, *Ann. de Math.*, 13 (1882), 258–267; Gregory, *Cambridge Math. Jour.*, 1 (1839), 54; Boole, "A Treatise on the Calculus of Finite Differences" (1860); Walton, *Quart. Jour.*, *10* (1870), 248–253; Combescure, *Ann. Ec. Nor. Sup.* (2), 3, (1874), 305–362; Cesàro, *Nouv. Ann.* (3), 4, (1885), 36–41; Laurent, "Traité de Analyse" (1890), Vol. 6, 234–236; Lemeray, *Edinburgh Math. Soc. Proc.* (1898), 13–14; Lecornu, *Bull. de Soc. Math. de France*, 27, (1899), 153–160; Oltramare, *Assoc. Fr. Marseille*, 20 (1891), 66–82; Oltramare, *Bordeaux Assoc. Fr. Bull.*, 24 (1899), 175–186; Oltramare, "Calcul de Generalisation" (1899); Brajtzew, Moscow Coll. (1901); Pincherle, *Rendiconti Pal.*, 18 (1904); Pincherle, *Mem. Soc. Italiana d. Sc.* (3), 15 (1907); Meissner, *Schweiz. Bauzeitung.*, 54; Polussuchin, Zurich Diss. (1910); Schmidt, *Math. Ann.*, 70 (1911), 499–524; Bateman, Proc. 5th Int. Cong. Math., Vol. 1 (1912), 291–294; Haag, *Bull. de Soc. Math.*, 36 (1912), 10–24; Schurer, *Ber. Gew. Wiss. Leipzig* (1912), 167–236; (1913), 139–143; (1914), 137–158; Carmichael, *Am. Jour.*, 35 (1913), 151–162; Bennett, *Ann. of Math.* (2), 18.

the elementary theory of the equation is extended along lines initiated by Poisson. Most of Poisson's results are incidentally included, but the work is from a different point of view, and the formulas obtained are more explicit, since their explicit forms are needed in the development of our further results. The theory of the invariants under the group of transformations $f(x) = v(x)g(x)$ is developed along the same lines as is the corresponding theory of the Laplace equation.* Largely the same methods are used, the analogy being very close. The results are summarized below by sections.

I. The functions

$$\frac{m(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)}, \qquad \text{and} \qquad \frac{m(x)}{p(x)} - q(x-1)$$

form a fundamental set of invariants under the group of transformations $f(x) = v(x)g(x)$, which transformations do not change the form of the equation.

II. When one of the fundamental invariants is zero the equation is of such nature that it may be obtained by differentiating a difference equation or else by applying the difference operation to a differential equation. That is, it may be solved by integrating first a linear differential [difference] equation and then a linear difference [differential] equation. These solutions each involve an arbitrary constant and an arbitrary periodic function of period one.

III. The Laplace-Poisson transformations

$$(S)\ f_{S_1}(x) = f(x+1) + p(x)f(x) \ \text{ and } \ (T)\ f_{T_1}(x) = f'(x) + q(x-1)f(x)$$

leave the form of the equation unchanged. The invariants of the equation gotten by applying S or T are simply expressible in terms of the invariants of the original equation. The two transformations are, in a sense, inverses of each other; for the application of both in either order to (1) gives an equation with the same invariants as (1). Successive applications of S, or of T, give equations of the same type, whose invariants can be expressed in terms of the invariants of the preceding equations under the successive transformations, and therefore in terms of the invariants of the original equation.

IV. The solutions of the equations obtained by successive applications of S or T may be obtained in terms of the solution of the nth transformed equation. In particular, the solution of the original equation may be thus obtained.

V. The term *rank* of the equation is introduced in accordance with the nomenclature of the corresponding theory of the Laplace partial differential

* An exposition of the theory of the partial differential equation $s + ap + bq + cz = 0$ may be found in Forsyth's "Theory of Differential Equations," Vol. 6, pp. 44–96.

equation.* The mixed equation is said to be of finite rank when a finite number of applications of S, or of T, results in an equation with a vanishing invariant. The equation can then be solved in finite form, and an arbitrary part of the solution can be so chosen that quadratures of arbitrary functions are not involved. The equation is said to be of rank $n+1$ of the first kind when n applications of S give an equation with a vanishing invariant. This is a necessary and sufficient condition for a solution of the form

$$f(x) = E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_n(x)F^{(n)}(x),$$

where the E's are determinate functions and $F(x)$ is an arbitrary periodic function of period one. The mixed equation is said to be of rank $n+1$ of the second kind when n applications of T give an equation with a vanishing invariant. In this case, the solution without quadrature of arbitrary functions is a determinate function of x multiplied by an arbitrary constant.

VI, VII. The restrictions on the coefficients of the mixed equation in order that it be of finite rank of the first kind or of the second kind are found.

VIII. When the equation is of finite rank with respect to both S and T, it is said to be of doubly finite rank. The restrictions on the coefficients of such an equation are found.

IX. Generalizations of the Laplace-Poisson transformations analogous to the transformations used by Lévy† in connection with the Laplace differential equation are here tried with a result similar to that found by Lévy, viz: that they are not generally useful in obtaining an equation of the type (1) with a vanishing invariant.‡

I. The Invariants of the Equation.

The equation§

$$(1)\qquad f'(x+1) + p(x)f'(x) + q(x)f(x+1) + m(x)f(x) = 0$$

is put by a transformation of the group

$$f(x) = v(x)g(x)$$

into the form

$$(2)\qquad g'(x+1) + P(x)g'(x) + Q(x)g(x+1) + M(x)g(x) = 0,$$

* Forsyth, l. c., p. 60.

† *Journal de l'École Polytechnique*, t. 38 (1886), p. 67.

‡ See Forsyth, l.c., p. 94.

§ We shall develop the theory only for the case when $p(x)$ is not zero. When $p(x) = 0$, $I(x)$ and $J(x)$ are illusory and $\alpha(x)$ and $\beta(x)$ are each equal to $m(x)$. It may be readily seen by following through this paper that the case $p(x) = 0$ can be carried, but the conditions for solutions here developed become simply $m(x) = 0$ when $p(x) = 0$, and the equation is then not a true mixed equation.

where

$$P(x) = p(x)\frac{v(x)}{v(x+1)}, \qquad Q(x) = \frac{v'(x+1)}{v(x+1)} + q(x),$$

and

$$M(x) = m(x)\frac{v(x)}{v(x+1)} + p(x)\frac{v'(x)}{v(x+1)}.$$

The form of the equation is therefore unaltered by the substitution. By eliminating $v(x)$ in two ways from the relations between the coefficients of (1) and (2), we may obtain

$$p(x)[M(x) - P(x)Q(x) - P'(x)] = P(x)[m(x) - p(x)q(x) - p'(x)].$$

and

$$p(x)[M(x) - P(x)Q(x-1)] = P(x)[m(x) - p(x)q(x)1)].$$

Hence

$$I(x) = \frac{m(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)}$$

and

$$J(x) = \frac{m(x)}{p(x)} - q(x-1)$$

are absolute invariants of the equation (1) under the group of transformations $f(x) = v(x)g(x)$. We shall find it convenient to use also the relative invariants

$$\alpha(x) = m(x) - p(x)q(x) - p'(x) \qquad \text{and} \qquad \beta(x) = m(x) - p(x)q(x-1).$$

These functions are each multiplied by $v(x)/v(x+1)$ at each application of $f(x) = v(x)g(x)$.

We will now show that $I(x)$ and $J(x)$ form a fundamental set of invariants of the equation (1); i.e., that all invariants of (1) under $f(x) = v(x)g(x)$ can be expressed as functions of $I(x)$ and $J(x)$ involving only algebraic operations, the operations of the differential calculus and the difference calculus and their inverses.

We will choose $v(x)$ in the transformation $f(x) = v(x)g(x)$ so that the equation (1) will be put into the form (2) subject to the restriction

$$P(x)Q(x) = M(x).$$

This condition reduces to

$$\frac{v(x)}{v(x+1)}[p(x)q(x) - m(x)] = p(x)\frac{d}{dx}\left[\frac{v(x)}{v(x+1)}\right],$$

whence we may take

$$\frac{v(x)}{v(x+1)} = e^{\int_{x_0}^{x}\frac{p(x)q(x)-m(x)}{p(x)}dx}.$$

This condition is also sufficient. So, by choosing $v(x)$ properly, we can transform the equation (1) into the form

$$g'(x+1) + P(x)g'(x) + Q(x)g(x+1) + P(x)Q(x)g(x) = 0.$$

The I and J invariants of this equation are

$$I(x) = -\frac{P'(x)}{P(x)}$$

and

$$J(x) = Q(x) - Q(x-1) = \Delta Q(x-1).$$

Accordingly

$$P(x) = ce^{-\int_{x_0}^{x} I(x)\,dx},$$

and

$$Q(x) = \Sigma J(x+1),$$

where Σ denotes some particular finite integral. Hence the transformed equation is

$$(3) \quad g'(x+1) + e^{-\int_{x_0}^{x} I(x)\,dx} g'(x) + \Sigma J(x+1)g(x+1) \\ + e^{-\int_{x_0}^{x} I(x)\,dx} \Sigma J(x+1)g(x) = 0.$$

This is of the same form as the original equation (1), and is derived from (1) by a transformation of the group $f(x) = v(x)g(x)$. Therefore (3) has the same invariants as (1) under transformations of the type considered. Since the invariants are functions of the coefficients alone, it follows that all the invariants of (3) are expressible in terms of $I(x)$ and $J(x)$ only. We shall refer to $I(x)$ and $J(x)$ simply as the invariants of the equation.

II. Solutions when One Invariant is Zero.

If $I(x) = 0$, then $\alpha(x) = 0$,

and

$$m(x) = p'(x) + p(x)q(x).$$

The equation may then be written in the form

$$\frac{d}{dx}[f(x+1) + p(x)f(x)] + q(x)[f(x+1) + p(x)f(x)] = 0,$$

whence

$$f(x+1) + p(x)f(x) = ce^{-\int_{x_0}^{x} q(x)\,dx}.$$

To solve this, we first solve the homogeneous equation

$$g(x+1) - [-p(x)]g(x) = 0$$

as follows

$$\log g(x+1) - \log g(x) = \log[-p(x)]$$

or

$$g(x) = \varphi(x)e^{\Sigma \log[-p(x)]},$$

where $\varphi(x)$ is an arbitrary periodic function of period one, and Σ denotes a finite integral* for some range of the variable x.

Let $f(x) = u(x)g(x)$ and substitute in the non-homogeneous equation. We get, taking $\varphi(x) = 1$,

$$u(x+1)e^{\Sigma \log[-p(x+1)]} - [-p(x)]u(x)e^{\Sigma \log[-p(x)]} = ce^{-\int_{x_0}^{x} q(x)dx}.$$

Hence

$$u(x+1) - u(x) = ce^{-\int_{x_0}^{x} q(x)dx - \Sigma \log[-p(x+1)]},$$

and therefore

$$u(x) = \Sigma\left[ce^{-\int_{x_0}^{x} q(x)dx - \Sigma \log[-p(x+1)]}\right] + F(x)$$

where $F(x)$ has the period one and is otherwise arbitrary. So we have

$$(4) \qquad f(x) = e^{\Sigma \log[-p(x)]}\left\{F(x) + \Sigma ce^{-\int_{x_0}^{x} q(x)dx - \Sigma \log[-p(x+1)]}\right\}.$$

If $J(x) = 0$, then $\beta(x) = 0$, and

$$m(x) = p(x)q(x-1).$$

The equation may then be written

$$f'(x+1) + q(x)f(x+1) + p(x)[f'(x) + q(x-1)f(x)] = 0,$$

from which we obtain†

$$\Delta \log[f'(x) + q(x-1)f(x)] = \log[-p(x)],$$

whence

$$f'(x) + q(x-1)f(x) = \theta(x)e^{\Sigma \log[-p(x)]},$$

$\theta(x)$ being an arbitrary periodic function of period one. Hence we have

$$(5) \quad f(x) = e^{-\int_{x_0}^{x} q(x-1)dx} \int_{x_0}^{x} \theta(x)e^{\Sigma \log[-p(x)] + \int_{x_0}^{x} q(x-1)dx} + ke^{-\int_{x_0}^{x} q(x-1)dx},$$

where K is an arbitrary constant.

* $F(x)$ is said to be a finite integral of $G(x)$ if

$$F(x+1) - F(x) = G(x).$$

In this paper, the symbol Σ without limits of summation denotes a finite integral. When used with limits, e.g., $\sum_{k=0}^{n}$, it denotes an ordinary summation.

† Δ denotes the difference of a function, i.e.,

$$\Delta v(x) = v(x+1) - v(x).$$

We have thus shown that when $I(x) = 0$ [$J(x) = 0$] a solution of the equation (1) can be obtained in finite form by solving, first a linear differential [difference] equation of the first order, and then a linear difference [differential] equation of the first order.

III. The Laplace-Poisson Transformations and the Invariants of the Resulting Equation.

The Laplace-Poisson transformation

$$(S) \qquad F_{S_1}(x) = f(x+1) + p(x)f(x)$$

transforms the equation (1) into an equation of the same form, viz:

$$(6) \qquad f'_{S_1}(x+1) + p_{S_1}(x)f'_{S_1}(x) + q_{S_1}(x)f_{S_1}(x+1) + m_{S_1}(x)f_{S_1}(x) = 0,$$

where

$$p_{S_1}(x) = p(x)\frac{\alpha(x+1)}{\alpha(x)} = p(x+1)\frac{I(x+1)}{I(x)},$$

$$q_{S_1}(x) = q(x+1),$$

and

$$m_{S_1}(x) = p(x+1)q(x)\frac{I(x+1)}{I(x)} + p(x+1)I(x+1).$$

The invariants of (6) under the group $f_{S_1}(x) = v(x)g(x)$ are

$$\begin{aligned} J_{S_1}(x) &= \frac{m_{S_1}(x)}{p_{S_1}(x)} - q_{S_1}(x-1) \\ &= q(x) + I(x) - q(x) \\ &= I(x). \end{aligned}$$

and

$$\begin{aligned} I(x) &= \frac{m_{S_1}(x)}{p_{S_1}(x)} - q_{S_1}(x) - \frac{p'_{S_1}(x)}{p_{S_1}(x)} \\ &= q(x) + I(x) - q(x+1) - \frac{p'(x+1)}{p(x+1)} - \Delta\frac{d}{dx}\log I(x). \end{aligned}$$

If we add and subtract $m(x+1)/p(x+1)$, we get

$$I_{S_1}(x) = I(x+1) + I(x) - J(x+1) - \Delta\frac{d}{dx}\log I(x).$$

Under the Laplace-Poisson transformation

$$(T) \qquad f_{T_1}(x) = f'(x) + q(x-1)f(x)$$

the equation (1) becomes

$$(7) \qquad f'_{T_1}(x+1) + p_{T_1}(x)f'_{T_1}(x) + q_{T_1}(x)f_{T_1}(x+1) + m_{T_1}(x)f_{T_1}(x) = 0,$$

in which the coefficients may be reduced to the following forms:

$$p_{T_1}(x) = p(x),$$

$$q_{T_1}(x) = q(x-1) - \frac{J'(x)}{J(x)} - \frac{p'(x)}{p(x)},$$

and

$$m_{T_1}(x) = p(x)J(x) + p(x)q(x-1) - p(x)\frac{J'(x)}{J(x)}.$$

The invariants of (7) under the group $f(x)\ v(x)g(x)$ are

$$\begin{aligned} I_{T_1}(x) &= \frac{m_{T_1}(x)}{p_{T_1}(x)} - q_{T_1}(x) - \frac{p'_{T_1}(x)}{p_{T_1}(x)} \\ &= J(x) + q(x-1) - \frac{J'(x)}{J(x)} - q(x-1) + \frac{p'(x)}{p(x)} + \frac{J'(x)}{J(x)} - \frac{p'(x)}{p(x)} \\ &= J(x) \end{aligned}$$

and

$$\begin{aligned} J_{T_1}(x) &= \frac{m_{T_1}(x)}{p_{T_1}(x)} - q_{T_1}(x-1) \\ &= J(x) + q(x-1) - \frac{J'(x)}{J(x)} + \frac{J'(x-1)}{J(x-1)} + \frac{p'(x-1)}{p(x-1)} - q(x-2). \end{aligned}$$

If we add and subtract $m(x-1)/p(x-1)$, we get

$$J_{T_1}(x) = J(x) + J(x-1) - I(x-1) - \Delta\frac{d}{dx}\log J(x-1).$$

Hence we see that the transformations S and T each transform the equation (1) into an equation of the same form, the invariants of which may be simply expressed in terms of the invariants of (1).

The two transformations S and T are, in a sense, inverses of each other; for TS gives

$$\begin{aligned} f_{TS}(x) &= f'_{S_1}(x) + q(x)f_{S_1}(x) \\ &= f'(x+1) + p(x)f'(x) + p'(x)f(x) + q(x)f(x+1) + p(x)q(x)f(x) \\ &= [p'(x) + p(x)q(x) - m(x)]f(x) \\ &= -p(x)I(x)f(x) = -\alpha(x)f(x), \end{aligned}$$

and ST gives

$$\begin{aligned} f_{ST}(x) &= f_{T_1}(x+1) + p(x)f_{T_1}(x) \\ &= f'(x+1) + q(x)f(x+1) + p(x)f'(x) + p(x)q(x-1)f(x) \\ &= [p(x)q(x-1) - m(x)]f(x) \\ &= -p(x)J(x)f(x) = -\beta(x)f(x). \end{aligned}$$

Hence the equations resulting from applications of TS and ST have the same invariants as has the original equation. Furthermore we may transform the equation (1) into itself as follows. Apply $TS[ST]$. The resulting equation is that obtained by replacing $f(x)$ in (1) by $\alpha(x)f(x)[\beta(x)f(x)]$. Then the transformation $f(x) = g(x)/\alpha(x)[f(x) = g(x)/\beta(x)]$ brings us back to the equation (1).

Let $I_{S_n}(x)$ and $J_{S_n}(x)$ be the invariants of the equation obtained by n successive applications of S. Then we have

$$J_{S_n}(x) = I_{S_{n-1}}(x),$$

and

$$I_{S_n}(x) = I_{S_{n-1}}(x+1) + I_{S_{n-1}}(x) - J_{S_{n-1}}(x) - \Delta \frac{d}{dx} \log I_{S_{n-1}}(x).$$

So we can write

$$I_{S_n}(x) - I_{S_{n-1}}(x+1) - I_{S_{n-1}}(x) + I_{S_{n-2}}(x+1) = -\Delta \frac{d}{dx} \log I_{S_{n-1}}(x)$$

$$I_{S_{n-1}}(x) - I_{S_{n-2}}(x+1) - I_{S_{n-2}}(x) + I_{S_{n-3}}(x+1) = -\Delta \frac{d}{dx} \log I_{S_{n-2}}(x)$$

. . . .

. . . .

$$I_{S_1}(x) - I(x+1) - I(x) + J(x+1) = -\Delta \frac{d}{dx} \log I(x).$$

Adding these, we get

$$I_{S_n}(x) - I_{S_{n-1}}(x+1) = I(x) - J(x+1) - \Delta \frac{d}{dx} \log [I(x) I_{S_1}(x) \cdots I_{S_{n-1}}(x)].$$

Write this for $n, n-1, n-2, \cdots$ successively, and add 1 to the argument at each step. We then have

$$I_{S_n}(x) - I_{S_{n-1}}(x+1) = I(x) - J(x+1) - \Delta \frac{d}{dx} \log [I(x) I_{S_1}(x) \cdots I_{S_{n-1}}(x)]$$

$$I_{S_{n-1}}(x+1) - I_{S_{n-2}}(x+2)$$
$$= I(x+1) - J(x+2) - \Delta \frac{d}{dx} \log [I(x+1) \cdots I_{S_{n-2}}(x+1)]$$

. . . .

. . . .

$$I_{S_1}(x+n-1) - I(x+n)$$
$$= I(x+n-1) - J(x+n) - \Delta \frac{d}{dx} \log I(x+n-1).$$

Adding, we get

$$I_{S_n}(x) - I(x+n) = \sum_{k=0}^{n-1} I(x+k) - \sum_{k=1}^{n} J(x+k)$$
$$- \Delta \frac{d}{dx} \log \left[\prod_{k=0}^{n-1} I(x+k) \prod_{k=0}^{n-2} I_{S_1}(x+k) \cdots I_{S_{n-1}}(x) \right],$$

or

$$(8) \quad I_{S_n}(x) = I(x) + \sum_{k=1}^{n} [I(x+k) - J(x+k)]$$
$$- \Delta \frac{d}{dx} \log \left[\prod_{k=0}^{n-1} I(x+k) \prod_{k=0}^{n-2} I_{S_1}(x+k) \prod_{k=0}^{n-3} I(x+k) \cdots I_{S_{n-1}}(x) \right].$$

Also

$$(9) \quad J_S(x) = I_{S_{n-1}}(x) = I(x) + \sum_{k=1}^{n-1} [I(x+k) - J(x+k)]$$
$$- \Delta \frac{d}{dx} \log \left[\prod_{k=0}^{n-2} I(x+k) \prod_{k=0}^{n-3} I_{S_1}(x+k) \cdots I_{S_{n-2}}(x) \right].$$

If $I_{T_n}(x)$ and $J_{T_n}(x)$ are the invariants of the equation obtained by applying T n times, we have

$$I_{T_n} = J_{T_{n-1}}(x),$$

and

$$J_{T_n}(x) = J_{T_{n-1}}(x) + J_{T_{n-1}}(x-1) - I_{T_{n-1}}(x-1) - \Delta \frac{d}{dx} \log J_{T_{n-1}}(x-1).$$

So we may write

$$J_{T_n}(x) - J_{T_{n-1}}(x-1) - J_{T_{n-1}}(x) + J_{T_{n-2}}(x-1) = -\Delta \frac{d}{dx} \log J_{T_{n-1}}(x-1)$$

$$J_{T_{n-1}}(x) - J_{T_{n-2}}(x-1) - J_{T_{n-2}}(x) + J_{T_{n-3}}(x-1) = -\Delta \frac{d}{dx} \log J_{T_{n-2}}(x-1)$$

. . . .

. . . .

$$J_{T_1}(x) - J(x-1) - J(x) + I(x-1) = -\Delta \frac{d}{dx} \log J(x-1).$$

Adding these, we get

$$J_{T_n}(x) - J_{T_{n-1}}(x-1) = J(x) - I(x-1) - \Delta \frac{d}{dx} \log [J(x-1) J_{T_1}(x-1) \cdots J_{T_{n-1}}(x-1)].$$

Write this for $n, n-1, n-2, \cdots$ successively, subtracting 1 from the argument at each step. We have then

$$J_{T_n}(x) - J_{T_{n-1}}(x-1) = J(x) - I(x-1) - \Delta \frac{d}{dx} \log [J(x-1) J_{T_1}(x-1) \cdots J_{T_{n-1}}(x-1)]$$

$$J_{T_{n-1}}(x-1) - J_{T_{n-2}}(x-2) = J(x-1) - I(x-2) - \Delta \frac{d}{dx} \log [J(x-2) J_{T_1}(x-2) \cdots J_{T_{n-2}}(x-2)]$$

. . . .

. . . .

$$J_{T_1}(x-n+1) - J(x-n) = J(x-n+1) - I(x-n) - \Delta \frac{d}{dx} \log [J(x-n)].$$

Adding these, we get

$$J_{T_n}(x) - J(x-n) = \sum_{k=0}^{n-1} J(x-k) - \sum_{k=1}^{n} I(x-k) - \Delta \frac{d}{dx} \log \left[\prod_{k=1}^{n} J(x-k) \prod_{k=1}^{n-1} J_{T_1}(x-k) \cdots J_{T_{n-1}}(x-1) \right],$$

and therefore

$$J_{T_n}(x) = J(x) + \sum_{k=1}^{n} [J(x-k) - I(x-k)] - \Delta \frac{d}{dx} \log \left[\prod_{k=1}^{n} J(x-k) \prod_{k=1}^{n-1} J_{T_1}(x-k) \cdots J_{T_{n-1}}(x-1) \right]. \tag{10}$$

Also

(11) $$I_{T_n}(x) = J_{T_{n-1}}(x) = J(x) + \sum_{k=1}^{n-1}[J(x-k) - I(x-k)]$$
$$- \Delta \frac{d}{dx}\log\left[\prod_{k=1}^{n-1} J(x-k) \prod_{k=1}^{n-2} J_{T_1}(x-k) \cdots J_{T_{n-2}}(x-1)\right].$$

So we have the result that after n successive transformations of the equation (1) by $S[T]$, we arrive at an equation whose invariants can be expressed explicitly in terms of the invariants of (1) and the invariants of the intermediate equations obtained by 1, 2, 3, $\cdots$, $n-1$ applications of $S[T]$, and hence in terms of the invariants of the original equation.

IV. Solutions of Successively Transformed Equations.

After $n+1$ applications of S the new dependent variable is $f_{S_{n+1}}(x)$. Operate with T and call the resulting dependent variable $f_{S_{n+1},T_1}(x)$. We have then

$$f_{S_{n+1}}(x) = f_{S_n}(x+1) + p_{S_n}(x) f_{S_n}(x),$$
$$f_{S_{n+1},T_1}(x) = f'_{S_{n+1}}(x) + q_{S_{n+1}}(x-1) f_{S_{n+1}}(x)$$
$$= - p_{S_n}(x) I_{S_n}(x) f_{S_n}(x).$$

Multiplying by $\exp\left[\int_{x_0}^{x} q(x+n)dx\right]$, and remembering that

$$q_{S_{n+1}}(x-1) = q(x+n),$$

we have

$$\frac{d}{dx}\left[f_{S_{n+1}}(x) e^{\int_{x_0}^{x} q(x+n)dx}\right] = - p_{S_n}(x) I_{S_n}(x) f_{S_n}(x) e^{\int_{x_0}^{x} q(x+n)dx},$$

and therefore

$$f_{S_n}(x) e^{\int_{x_0}^{x} q(x+n)dx} = \frac{-1}{p_{S_n}(x) I_{S_n}(x)} \frac{d}{dx}\left[f_{S_{n+1}}(x) e^{\int_{x_0}^{x} q(x+n)dx}\right].$$

Since

$$e^{\int_{x_0}^{x} q(x+n)dx} = e^{\int_{x_0}^{x}[q(x+n)-q(x+n-1)+q(x+n-1)]dx}$$
$$= e^{\int_{x_0}^{x} q(x+n-1)dx}\, e^{\int_{x_0}^{x} \Delta q(x+n-1)dx}.$$

We may write

$$f_{S_n}(x) e^{\int_{x_0}^{x} q(x+n-1)dx} = A_n \frac{dB_n}{dx},$$

where

$$A_n = \frac{e^{\int_{x_0}^{x} \Delta q(x+n-1)dx}}{-p_{S_n}(x) I_{S_n}(x)}$$

and

$$B_n = f_{S_{n+1}}(x) e^{\int_{x_0}^{x} q(x+n)dx}.$$

Then we have

$$
(12)\quad
\begin{aligned}
f_{S_{n-1}}(x)e^{\int_{x_0}^{x} q(x+n-2)\,dx} &= A_{n-1}\frac{d}{dx}\left(A_n\frac{d}{dx}B_n\right),\\
f_{S_{n-2}}(x)e^{\int_{x_0}^{x} q(x+n-3)\,dx} &= A_n\frac{d}{dx}\left[A_{n-1}\frac{d}{dx}\left(A_n\frac{d}{dx}B_n\right)\right],\\
&\cdots\cdots\cdots\cdots\\
f(x)e^{\int_{x_0}^{x} q(x-1)\,dx} &= A_0\frac{d}{dx}\left\{A_1\frac{d}{dx}\left[\cdots\frac{d}{dx}\left(A_n\frac{d}{dx}B_n\right)\right]\right\},
\end{aligned}
$$

where the successive A's are gotten by replacing n in A_n by $n-1$, $n-2$, $n-3$, $\cdots$, 2, 1, 0, if we agree that $p_{S_0}(x) = p(x)$ and $I_{S_0}(x) = I(x)$.

In a similar way we can get an expression for $f(x)$ in terms of $f_{T_n}(x)$, the dependent variable after n applications of T. The $(n+1)$th application of T gives

$$f_{T_{n+1}}(x) = f_{T_n}(x) + q_{T_n}(x-1)f_{T_n}(x).$$

Then. operating with S, we get, since $p_{T_{n+1}}(x) = p(x)$,

$$
\begin{aligned}
f_{T_{n+1},S_1}(x) &= f_{T_{n+1}}(x+1) + p(x)f_{T_{n+1}}(x)\\
&= -\,p(x)J_{T_n}(x)f_{T_n}(x).
\end{aligned}
$$

We want to find a quantity $\chi(x)$ such that

$$\chi(x)[f_{T_{n+1}}(x+1) + p(x)f_{T_{n+1}}(x)] = \Delta[f_{T_{n+1}}(x)\psi(x)].$$

We may then take

$$\chi(x) = \psi(x+1),$$

and

$$p(x)V(x) = -\,\psi(x).$$

These two conditions give

$$\psi(x) = e^{-\Sigma\log[-p(x)]}.$$

Then we have

$$\chi(x) = \psi(x+1) = e^{-\Sigma\log[-p(x+1)]}.$$

So we write

$$[f_{T_{n+1}}(x+1) + p(x)f_{T_{n+1}}(x)]e^{-\Sigma\log[-p(x+1)]} = -\,p(x)J_{T_n}(x)e^{-\Sigma\log[-p(x+1)]}f_{T_n}(x),$$

whence

$$\Delta f_{T_{n+1}}(x)e^{-\Sigma\log[-p(x)]} = -\,p(x)J_{T_n}(x)f_{T_n}(x)e^{-\Sigma\log[-p(x+1)]},$$

and therefore

$$f_{T_n}(x)e^{-\Sigma\log[-p(x+1)]} = C_n\Delta D_n,$$

where

$$C_n = \frac{-1}{p(x)J_{T_n}(x)},$$

and

$$D_n = f_{T_{n+1}}(x)e^{-\Sigma \log[-p(x)]},$$

Then we have

$$f_{T_{n-1}}(x)e^{-\Sigma \log[-p(x+1)]} = C_{n-1}\Delta(C_n\Delta D_n)$$
$$f_{T_{n-2}}(x)e^{-\Sigma \log[-p(x+1)]} = C_{n-2}\Delta[C_{n-1}\Delta(C_n\Delta D_n)]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

(13) $$f(x)e^{-\Sigma \log[-p(x+1)]} = C_0\Delta\{C_1\Delta[\cdots\Delta(C_n\Delta D_n)]\},$$

where the C's are gotten by replacing n in C_n by $n-1, n-2, \cdots, 2, 1, 0$, if we agree that $J_{T_0}(x) = J(x)$.

Now we have expressed the solution $f(x)$ of the equation (1) in terms of $f_{S_n}(x)[f_{T_n}(x)]$, the solution of the nth transformed equation under $S[T]$. We have seen that we can find $f_{S_n}(x)[f_{T_n}(x)]$ if $I_{S_n}(x)=0[I_{T_n}(x)=0]$ or if $J_{S_n}(x) = 0$ $[J_{T_n}(x) = 0]$, and these solutions will be in finite form.

V. THE RANK OF THE EQUATION.

Suppose $I_{S_n}(x) = 0$. The equation may then be written in the form

$$\frac{d}{dx}[f_{S_n}(x+1) + p_{S_n}(x)f_{S_n}(x)] + q_{S_n}(x)[f_{S_n}(x+1) + p_{S_n}(x)f_{S_n}(x)] = 0,$$

which, as may be seen from § II, has a solution of the form

$$f_{S_n}(x) = e^{\Sigma \log[-p_{S_n}(x)]}\left\{F(x) + \Sigma c e^{-\int_{x_0}^{x} q_{S_n}(x)dx - \Sigma \log[-p_{S_n}(x+1)]}\right\},$$

where $F(x)$ has the period one and is otherwise arbitrary, and where c is an arbitrary constant.

As before, denote $e^{\int_{x_0}^{x} q(x+n-1)dx} f_{S_n}(x)$ by B_{n-1}. Then this expression is seen to be of the form

$$B_{n-1} = \eta(x)[F(x) + c\Sigma\xi(x)],$$

where $\eta(x)$ and $\xi(x)$ are determinate functions of x. Making use of a previously derived formula, viz:

(12) $$e^{\int_{x_0}^{x} q(x-1)dx} f(x) = A_0\frac{d}{dx}\left\{A_1\frac{d}{dx}\left[\cdots\frac{d}{dx}\left(A_n\frac{d}{dx}B_n\right)\right]\right\},$$

we get an expression for $f(x)$ of the form

$$\begin{aligned} f(x) = E_0(x)\{F(x) + c\Sigma\xi(x)\} &+ E_1(x)\{F'(x) + [c\Sigma\xi(x)]'\} \\ &+ E_2(x)[F''(x) + [c\Sigma\xi(x)]''\} + \cdots \\ &+ \cdots + \cdots \\ &+ \cdots + E_n(x)\{F^{(n)}(x) + [c\Sigma\xi(x)]^{(n)}\}, \end{aligned}$$

where the E's are determinate functions of x.

Taking the particular integral for which $c = 0$, we have a simpler integral which does not involve $\Sigma\xi(x)$, viz:

$$f(x) = E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_n(x)F^{(n)}(x). \tag{14}$$

We will call this an integral of *rank* $n+1$ of the *first kind.* We will call the original equation of rank $n+1$ of the first kind when $I_{S_n}(x) = 0$* and $I_{S_k}(x) \neq 0$, where k takes the values $0, 1, 2, \cdots, n-1$.

Conversely, if the original equation

$$f'(x+1) + p(x)f'(x) + q(x)f(x+1) + m(x)f(x) = 0 \tag{1}$$

has a solution of the form (14), where the E's are determinate functions of x, and $F(x)$ is an arbitrary periodic function of period one; then after at most n applications of the transformation S, we will have an equation of which the I invariant is zero, i.e.,

$$I_{S_\mu}(x) = 0, \qquad (\mu \leqq n).$$

To show this, substitute in (1) the expression for $f(x)$ in (14), remembering that

$$F(x+\mu) = F(x).$$

We then have

$$\begin{aligned} E_0'(x+1)F(x)+E_0(x+1)F'(x)+ \cdots +E_n'(x+1)F^{(n)}(x)+E_n(x+1)F^{(n+1)}(x) \\ + p(x)[E_0'(x)F(x)+E_0(x)F'(x)+ \cdots E_n'(x)F^{(n)}(x)+E_n(x)F^{(n+1)}(x)] \\ + q(x)[E_0(x+1)F(x)+E_1(x+1)F'(x)+ \cdots +E_n(x+1)F^{(n)}(x)] \\ +m(x)[E_0(x)F(x)+E_1(x)F'(x)+ \cdots +E_n(x)F^{(n)}(x)] = 0, \end{aligned}$$

which may be written in the form

$$K_{n+1}(x)F^{(n+1)}(x) + K_n(x)F^{(n)}(x) + K_{n-1}(x)F^{(n-1)}(x) + \cdots = 0,$$

where

$$\begin{aligned} K_{n+1}(x) &= E_n(x+1) + p(x)E_n(x), \\ K_n(x) &= E_{n-1}(x+1) + p(x)E_{n-1}(x) + E_n'(x+1) + p(x)E_n'(x) \\ &\qquad + q(x)E_n(x+1) + m(x)E_n(x), \\ K_{n-1}(x) &= E_{n-2}(x+1) + p(x)E_{n-2}(x) + E_{n-1}'(x+1) + p(x)E_{n-1}'(x) \\ &\qquad + q(x)E_{n-1}(x+1) + m(x)E_{n-1}(x+1), \end{aligned}$$

and so forth. All of these K's must be zero since $f(x)$ satisfies the equation (1) for all values of $F(x)$. Hence

$$E_n(x+1) = -p(x)E_n(x),$$

and

$$\begin{aligned} E_{n-1}(x+1) = p(x)E_{n-1}(x) &= \frac{d}{dx}[p(x)E_n(x)] - p(x)E_n'(x) \\ &\qquad + p(x)q(x)E_n(x) - m(x)E_n(x) \\ &= p'(x)E_n(x) + p(x)q(x)E_n(x) - m(x)E_n(x) \\ &= -p(x)I(x)E_n(x). \end{aligned}$$

* Note that if $J_{S_n}(x) = 0$ then $I_{S_{n-1}}(x) = 0$, as may be seen by referring to § III.

Applying the transformation S to $f(x)$, we have

$$\begin{aligned} f^S_1(x) &= f(x+1) + p(x)f(x) \\ &= E_0(x+1)F(x) + E_1(x+1)F'(x) + \cdots + E_n(x+1)F^{(n)}(x) \\ &\qquad + p(x)[E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_n(x)F^{(n)}(x)] \\ &= [E_n(x+1) + p(x)E_n(x)]F^{(n)}(x) \\ &\qquad + [E_{n-1}(x+1) + p(x)E_{n-1}(x)]F^{(n-1)}(x) + \cdots \\ &= 0 - p(x)I(x)F^{(n+1)}(x) + \cdots. \end{aligned}$$

We see that the order of the transformed expression in $F(x)$ is less than before. Repeat the process, reducing the order each time, until we get one of the invariants $I_k(x)$ zero, or else we get a new dependent variable

$$f_{S_\mu}(x) = R(x)F(x)$$

which satisfies the equation

$$f'_{S_\mu}(x+1) + p_{S_\mu}(x)f'_{S_\mu}(x) + q_{S_\mu}(x)f_{S_\mu}(x+1) + m_{S_\mu}(x)f_{S_\mu}(x) = 0,$$

or

$$\begin{aligned} R'(x+1)F(x) + R(x+1)F'(x) + p_{S_\mu}(x)[R'(x)F(x) + R(x)F'(x)] \\ + q_{S_\mu}(x)R(x+1)F(x) + m_{S_\mu}(x)R(x)F(x) = 0. \end{aligned}$$

This equation is an identity in $F(x)$, so the coefficients of $F(x)$ and of $F'(x)$ must be zero. Setting the coefficient of $F'(x)$ equal to zero, we get

$$p_{S_\mu}(x) = -\frac{R(x+1)}{R(x)}.$$

Putting this into the coefficient of $F(x)$ set equal to zero, we have

$$R'(x+1) - \frac{R(x+1)R'(x)}{R(x)} + q_{S_\mu}(x)R(x+1) + m_{S_\mu}(x)R(x) = 0.$$

Forming the invariant $I_{S_\mu}(x)$, we have

$$\begin{aligned} p(x)I_{S_\mu}(x) &= m_{S_\mu}(x) - p_{S_\mu}(x)q(x) - p'_{S_\mu}(x) \\ &= \frac{R(x+1)R'(x)}{R^2(x)} - \frac{R'(x+1)}{R(x)} - q_{S_\mu}(x)\frac{R(x+1)}{R(x)} \\ &\qquad + q_{S_\mu}(x)\frac{R(x+1)}{R(x)} + \frac{R(x)R'(x+1) - R(x+1)R'(x)}{R(x)} \end{aligned}$$

which is identically zero. Hence we see that $I_{S_\mu}(x) = 0$ is a necessary as well as a sufficient condition for a solution of the form

$$f(x) = E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_n(x)F^{(n)}(x). \tag{14}$$

Suppose that $J_{T_n}(x) = 0$* and $J_{T_k}(x) \neq 0$, where k takes the values $0, 1, 2, 3, \cdots, n-1$. The equation in $f_{T_n}(x)$ can then be written

* Note that if $I_{T_n}(x) = 0$, then $J_{T_{n-1}}(x) = 0$, as may be seen by referring to § III.

$$f'_{T_n}(x+1) + q_{T_n}(x) f_{T_n}(x+1) - [-p(x)][f'_{T_n}(x) + q_{T_n}(x) f_{T_n}(x)] = 0,$$

which, as was shown in § II, has a solution of the form

$$f_{T_n}(x) = e^{-\int_{x_0}^{x} q_{T_n}(x-1)dx} \int_{x_0}^{x} \theta(x) e^{\Sigma \log[-p(x)] + \int_{x_0}^{x} q_{T_n}(x-1)dx} + K e^{-\int_{x_0}^{x} q_{T_n}(x-1)dx},$$

where $\theta(x)$ is an arbitrary function of period one, and K is an arbitrary constant. We will write for the sake of brief notation

$$e^{-\Sigma \log[-p(x)]} f_{T_n}(x) = e^{\lambda(x)} V(x).$$

We have already developed the formula

$$(13) \qquad e^{-\Sigma \log[-p(x)]} f(x) = C_0 \Delta \{ C_1 \Delta [\cdots \Delta (C_n \Delta D_n)] \}.$$

In our present case

$$D_n = e^{\lambda(x)} V(x) \qquad \text{and} \qquad C_n = \frac{-1}{p(x) J_{T_{n-1}}(x)}.$$

Then $f(x)$ takes the form

$$(15) \quad f(x) = W_0(x) V(x) + W_1(x) V(x+1) + \cdots + W_n(x) V(x+n),$$

where the W's are determinate functions. Choosing $\theta(x) = 0$, we have the simpler integral

$$f(x) = K[W_0(x) + W_1(x) \cdots + W_n(x)].$$

or

$$f(x) = KW(x),$$

where $W(x)$ is a determinate function and K is an arbitrary constant.

The solution (15) we will call of rank $n+1$ of the *second kind*, and the equation for which $J_{T_n}(x) = 0$, and $J_{T_k}(x) \neq 0$, where k takes the values $0, 1, 2, 3, \cdots, n-1$, we will also call of rank $n+1$ of the second kind. In this paper, we will be concerned with the rank of the equation rather than with the rank of the solution.

VI. Equations of Finite Rank of the First Kind.

Suppose the equation (1) is of rank $n+1$ of the first kind. We then have

$$\alpha_{S_n}(x) = m_{S_n}(x) - p_{S_n}(x) q_{S_n}(x) - p'_{S_n}(x) = 0.$$

If $p_{S_n}(x)$ and $q_{S_n}(x)$ are chosen arbitrarily, $m_{S_n}(x)$ is defined by this equation. Using the expression for $m_{S_n}(x)$ thus defined, the invariant

$$J_{S_n}(x) = \frac{m_{S_n}(x)}{p_{S_n}(x)} - q_{S_n}(x)$$

becomes

$$J_{S_n}(x) = \Delta q_{S_n}(x) + \frac{d}{dx} \log p_{S_n}(x).$$

We have proved that

$$I_{S_{m+1}}(x) = I_{S_m}(x+1) + I_{S_m}(x) - J_{S_m}(x) - \Delta \frac{d}{dx} \log I_{S_m}(x),$$

and

$$J_{S_{m+1}}(x) = I_{S_m}(x),$$

from which we get

$$J_{S_{m-1}}(x) = J_{S_m}(x+1) + J_{S_m}(x) - I_{S_m}(x) - \Delta \frac{d}{dx} \log J_{S_m}(x).$$

Therefore, since $I_{S_n}(x) = 0$,

$$J_{S_{n-1}}(x) = J_{S_n}(x+1) + J_{S_n}(x) - \Delta \frac{d}{dx} \log J_{S_n}(x).$$

So we can now calculate the coefficients in backward succession.

Since $q_{S_{m+1}}(x) = q_{S_m}(x-1)$, we have

$$q(x) = q_{S_n}(x-n).$$

Also we have

$$p_{S_{n-1}}(x+1) = p_{S_n}(x) \frac{I_{S_{n-1}}(x)}{I_{S_{n-1}}(x+1)},$$

or

$$p_{S_{n-1}}(x+1) = p_{S_n}(x) \frac{J_{S_n}(x)}{J_{S_n}(x+1)},$$

also

$$\begin{aligned} p_{S_{n-2}}(x+2) &= p_{S_{n-1}}(x+1) \frac{J_{S_{n-1}}(x+1)}{J_{S_{n-1}}(x+2)} \\ &= p_{S_n}(x) \frac{J_{S_n}(x) J_{S_{n-1}}(x+1)}{J_{S_n}(x+1) J_{S_{n-1}}(x+2)}, \end{aligned}$$

. . . .

. . . .

$$p(x+n) = p_{S_n}(x) \frac{J_{S_n}(x) J_{S_{n-1}}(x+1) \cdots J_{S_1}(x+n-1)}{J_{S_n}(x+1) J_{S_{n-1}}(x+2) \cdots J_{S_1}(x+n)}.$$

Reducing the argument by n, we have

$$p(x) = p_{S_n}(x-n) \frac{J_{S_n}(x-n) J_{S_{n-1}}(x-n+1) \cdots J_{S_1}(x-1)}{J_{S_n}(x-n+1) J_{S_{n-1}}(x-n+2) \cdots J_{S_1}(x)}.$$

Also we have

$$\begin{aligned} m(x) &= p(x)q(x) + p'(x) + p(x)I(x) \\ &= p(x)q(x) + p'(x) + p(x)J_{S_1}(x), \end{aligned}$$

or we may use the form

$$m(x) = p(x)q(x-1) + p(x)J(x).$$

Thus we have developed the restrictions upon the coefficients of (1) which must exist if (1) is of finite rank of the first kind. That these conditions are also sufficient, may be seen by reversing the steps of the discussion.

VII. EQUATIONS OF FINITE RANK OF THE SECOND KIND.

Suppose the equation (1) is of rank $n+1$ of the second kind. Then we have

$$\beta_{T_n}(x) = m_{T_n}(x) - p_{T_n}(x)q_{T_n}(x-1) = 0.$$

Hence if $p_{T_n}(x)$ and $q_{T_n}(x)$ are arbitrarily given, $m_{T_n}(x)$ is determined. We have already found

$$J_{T_n}(x) = J_{T_{n-1}}(x) + J_{T_{n-1}}(x-1) - I_{T_{n-1}}(x-1) - \Delta\frac{d}{dx}\log I_{T_{n-1}}(x-1),$$

and

$$J_{T_{n-1}}(x) = I_{T_n}(x).$$

From these we get, remembering that $J_{T_n}(x) = 0$,

$$I_{T_n}(x) = J_{T_{n-1}}(x) = \frac{m_{T_n}(x)}{p_{T_n}(x)} - q_{T_n}(x) - \frac{p'_{T_n}(x)}{p_{T_n}(x)}$$
$$= -\Delta q_{T_n}(x-1) - \frac{p'_{T_n}(x)}{p_{T_n}(x)}.$$

Now we can calculate the coefficients in backward succession. We have seen that

$$p(x) = p_{T_n}(x).$$

We also have

$$q_{T_n}(x) = q_{T_{n-1}}(x-1) - \frac{d}{dx}\log\left[J_{T_{n-1}}(x)p(x)\right],$$

whence

$$q_{T_{n-1}}(x-1) = q_{T_n}(x) + \frac{d}{dx}\log\left[J_{T_{n-1}}(x)p(x)\right].$$

Hence it follows that

$$q_{T_n}(x-2) = q_{T_n}(x) + \frac{d}{dx}\log\left[J_{T_{n-1}}(x)J_{T_{n-2}}(x-1)p(x)p(x-1)\right],$$
$$q(x-3) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$q(x-n) = q_{T_n}(x) + \frac{d}{dx}\log\left[J_{T_{n-1}}(x)\cdots J(x-n)p(x)\cdots p(x-n)\right].$$

Increasing the argument by n, we have

$$q(x) = q_{T_n}(x+n) + \frac{d}{dx}\log\left[J_{T_{n-1}}(x+n)\cdots J(x)p(x+n)\cdots p(x)\right],$$

or

$$q(x) = q_{T_n}(x+n) + \frac{d}{dx}\log\left[I_{T_n}(x+n)\cdots I_{T_1}(x)p(x+n)\cdots p(x)\right].$$

To determine $m(x)$, we have

$$m(x) = p(x)q(x-1) + p(x)J(x),$$

or

$$m(x) = p(x)q(x-1) + p(x)I_{T_1}(x).$$

Thus we have found the restrictions on the coefficients of (1) which must hold if (1) is of finite rank of the second kind. By reversing the steps of the discussion we may see that these conditions are also sufficient. So we now have the necessary and sufficient conditions that (1) shall be of finite rank of either the first or the second kind.

VIII. Equations of Doubly Finite Rank.

The equation (1) is said to be of doubly finite rank when it is of finite rank with respect to both S and T. Suppose it is of rank $k+1$ with respect to T, and of finite rank with respect to S. Transforming it k times by T, we have, since $J_{T_k}(x) = 0$,

$$f'_{T_k}(x+1) + p(x)f'_{T_k}(x) + q_{T_k}(x)f_{T_k}(x+1) + p(x)q_{T_k}(x-1)f_{T_k}(x) = 0. \tag{16}$$

This equation is also of finite rank with respect to S. Suppose that rank is $r+1$. We wish to see what restrictions are then imposed upon the coefficients $p(x)$ and $q_{T_k}(x)$.

First, apply the transformation

$$f_{T_k}(x) = v(x)h(x),$$

which does not change the rank. We will choose $v(x)$ so that the coefficient of $h(x+1)$ in the new equation shall be zero. This requires that

$$\frac{v'(x+1)}{v(x+1)} = -\, q_{T_k}(x),$$

whence

$$v(x) = e^{-\int_{x_0}^{x} q_{T_k}(x-1)dx}.$$

The equation then becomes

$$h'(x+1) + p(x)\frac{v(x)}{v(x+1)}h'(x) = 0,$$

which may be written in the form

$$h'(x+1) + p(x)e^{\int_x^{x+1} q(x-1)dx}h'(x) = 0. \tag{17}$$

This equation, being of rank $r+1$ with respect to S, has a solution of the form

$$h(x) = E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_r(x)F^{(r)}(x),$$

where $F(x)$ is an arbitrary function of period one. Then $h'(x)$ has the form

$$h'(x) = Z_0(x)F(x) + Z_1(x)F'(x) + \cdots + Z_{r+1}(x)F^{(r+1)}(x),$$

where

$$Z_0(x) = E'_0(x), \qquad Z_1(x) = E_0(x) + E'_1(x), \qquad \ldots.$$

Substitute this value of $h'(x)$ in (17). Since it must satisfy (17) identically, we have the relations

$$Z_i(x+1) - R(x)Z_i(x) = 0, \tag{18}$$

where $R(x)$ is the negative of the coefficient of $h'(x)$ in (17).

Let $\bar{Z}(x)$ be any particular solution of (18). The other solutions may then be expressed in the form

$$Z_i(x) = w_i(x)\bar{Z}(x),$$

where the functions $w_i(x)$ have the period one. In particular

$$\begin{aligned}
Z_0(x) &= w_0(x)\bar{Z}(x) = E_0'(x),\\
Z_1(x) &= w_1(x)\bar{Z}(x) = E_0(x) + E_1'(x),\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
Z_r(x) &= w_r(x)\bar{Z}(x) = E_{r-1}(x) + E_r'(x),\\
Z_{r+1}(x) &= w_{r+1}(x)\bar{Z}(x) = E_r(x).
\end{aligned}$$

From these we get

$$\begin{aligned}
E_r(x) &= w_{r+1}(x)\bar{Z}(x),\\
E_{r-1}(x) &= w_r(x)\bar{Z}(x) - \frac{d}{dx}[w_{r+1}(x)\bar{Z}(x)],\\
E_{r-2}(x) &= w_{r-1}(x)\bar{Z}(x) - \frac{d}{dx}[w_r(x)\bar{Z}(x)] + \frac{d^2}{dx^2}[w_{r+1}(x)Z(x)],\\
E_{r-3}(x) &= w_{r-2}(x)\bar{Z}(x) - \frac{d}{dx}[w_{r-1}(x)\bar{Z}(x)] + \frac{d^2}{dx^2}[w_r(x)\bar{Z}(x)]\\
&\qquad - \frac{d^3}{dx^3}[w_{r+1}(x)\bar{Z}(x)]\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
E_0(x) &= w_1(x)\bar{Z}(x) - \frac{d}{dx}[w_2(x)\bar{Z}(x)] + \cdots\\
&\qquad + \cdots + (-1)^r\frac{d^r}{dx^r}[w_{r+1}(x)\bar{Z}(x)].
\end{aligned}$$

Since $w_0(x)\bar{Z}(x) = E_0'(x)$, we have

$$w_0(x)\bar{Z}(x) = \frac{d}{dx}[w_1(x)\bar{Z}(x)] - \cdots + (-1)^r\frac{d^{r+1}}{dx^{r+1}}[w_{r+1}(x)\bar{Z}(x)],$$

which may be written in the form

$$\bar{Z}^{(r+1)}(x) + \eta_1(x)\bar{Z}^{(r)}(x) + \cdots + \eta_{r+1}(x)\bar{Z}(x) = 0. \tag{19}$$

The steps in the foregoing discussion are reversible. Hence it follows that the necessary and sufficient condition that the equation (1) shall be

of doubly finite rank is that

$$p(x)e^{\int_x^{x+1} q_{T_k}(x-1)dx} = \frac{\overline{Z}(x+1)}{\overline{Z}(x)},$$

where $\overline{Z}(x)$ is a solution of (19), in which the coefficients $\eta_i(x)$ involve the arbitrary periodic functions $w_i(x)$ as shown above.

IX. THE ANALOGUES OF LÉVY'S TRANSFORMATIONS.

In this section we consider generalizations of the Laplace-Poisson transformations, and investigate their usefulness in obtaining equations with vanishing invariants. These transformations are analogous to those applied by Lévy* to the analogous partial differential equation. They are

$$G(x) = f(x+1) + [p(x) + \gamma(x)]f(x)$$

and

$$H(x) = f'(x) + [q(x-1) + \delta(x)]f(x).$$

As the results are of a negative character, we shall merely state them without proof.

In case of the first transformation, the transformed equation cannot have a vanishing J-invariant unless we have $J(x) = 0$ from the original equation. The transformed equation cannot have a vanishing I-invariant except in very special cases.

In case of the second transformation, the transformed equation cannot have an I-invariant equal to zero unless the original equation gives $I(x) = 0$, and the J-invariant of the transformed equation cannot be zero except in very special cases.

So we have the result that the two transformations investigated in this section are not generally useful in obtaining an equation with a vanishing invariant.

UNIVERSITY OF ILLINOIS,
May, 1918.

* *Journal de l'École Polytechnique*, t. 38 (1886), p. 67.

CHARACTERISTIC SUBGROUPS OF AN ABELIAN PRIME POWER GROUP.

By G. A. Miller.

§ 1. *Introduction.*

A subgroup which corresponds to itself in every possible automorphism of a given group is called a characteristic subgroup or an I-invariant subgroup. Some fundamental properties of the characteristic subgroups of any abelian group were studied by the writer of the present article in a paper published in volume 27 of the American Journal of Mathematics, 1905, pages 15–24. In particular, it was noted in this paper that besides the identity there is a certain characteristic subgroup, called the fundamental characteristic subgroup, which appears in every possible characteristic subgroup of an abelian group G whose order is of the form p^m, p being some prime number.

The present paper is devoted to a determination of various new properties of the characteristic subgroups of G. For the sake of clearness it seems desirable to explain here a few terms which are frequently employed. Two groups H_1 and H_2 are said to be complementary groups as regards a group G provided at least one H_1 of these two groups is an invariant subgroup of G while the other H_2 is simply isomorphic with the quotient group of G with respect to H_1. When G is abelian it is known that H_2 is simply isomorphic with at least one subgroup of G and hence it is convenient to speak of *complementary subgroups of G. Two invariant subgroups H_1 and H_2 are said to be complementary subgroups of a group G provided each of these subgroups is simply isomorphic with the quotient group of G with respect to the other.*

It should first be noted that if one of two subgroups of G is simply isomorphic with the quotient group of G with respect to the other the two subgroups are not necessarily complementary. For instance, every subgroup of order p is simply isomorphic with each of the quotient groups arising from the subgroups of index p, but when G involves λ distinct invariants the quotient groups arising from its subgroups of order p are of λ distinct types. Each of these types corresponds to one set of I-conjugate subgroups of order p. That is, there are just λ sets of I-conjugate complementary subgroups of order and of index p. It will be seen in the following section that these correspond to the λ characteristic subgroups generated by operators of order p, and the λ characteristic subgroups involving oper-

ators of order p^{α_1-1}, p^{α_1} being the order of the largest operators contained in G.

The complementary subgroup of the fundamental characteristic subgroup is composed of all the operators of G whose orders divide p^{α_1-1} and it is characterized by the fact that it is the only characteristic subgroup of G which includes every other characteristic subgroup of G. It is the cross-cut of all the subgroups of index p under G which are complementary to the subgroups of order p contained in the fundamental characteristic subgroup of G. It should be noted that the numbers of these complementary subgroups of order and of index p are equal to each other.

The simplest characteristic subgroups of G are those composed of all the operators of G whose orders divide p^β, $\beta < \alpha_1$. The complementary subgroup of such a characteristic subgroup is composed of the p^β power of every operator of G. A necessary and sufficient condition that there are no other characteristic subgroups in G is that all the invariants of G are equal to each other. In this case, the number of the characteristic subgroups, besides the identity, is therefore equal to $\alpha_1 - 1$, and the number of pairs of complementary characteristic subgroups is $(\alpha_1 - 1)/2$ when α_1 is odd. When α_1 is even, the number of these pairs is $(\alpha_1 - 2)/2$ and one of the characteristic subgroups is self-complementary.

It may be desirable to direct attention to the difference between complementary subgroups and subgroups of complementary types. If G is of type $(m_1, m_2, \cdots, m_\lambda)$, and if two of its subgroups are of types $(\alpha_1, \alpha_2, \cdots, \alpha_{\lambda_1})$ and $(\beta_1, \beta_2, \cdots, \beta_{\lambda_2})$ respectively, these subgroups are said to be of complementary types when it is possible to satisfy each of the following equations, where x is either 0 or some α, and y is either 0 or some β and each α or β is used only once*:

$$x + y = m_i \qquad (i = 1, 2, \cdots, \lambda).$$

Subgroups which are of complementary types are clearly also complementary subgroups. That the converse is not necessarily true may be seen by considering the group of type (4, 1). This group contains operators of order p^3 which are not powers of operators of order p^4 and such an operator of order p^3 generates a subgroup whose complementary subgroups are cyclic and of order of p^2 but are not contained separately in cyclic groups of order p^4. Hence $\alpha_1 = \alpha_{\lambda_1} = 3$ and $\beta_1 = \beta_{\lambda_2} = 2$ in the present case, so that neither of the two equations $x + y = m_i$, $(i = 1, 2)$, can be satisfied. The complementary subgroup of the cyclic subgroup of order p^3 which is contained in the cyclic subgroups of order p^4 is of type (1, 1), and in this case the complementary subgroups are also of complementary types.

* G. A. Miller, *Transactions of the American Mathematical Society*, vol. 21 (1920), p. 313.

§ 2. *Characteristic subgroups generated by operators of a given order.*

The number of the characteristic subgroups contained in G depends on the number of the different invariants of G but is not affected by the number of these invariants which are equal to each other. That is, if no two of the invariants of G are equal to each other G has exactly the same number of characteristic subgroups as the group G which includes G and has no invariant except such as are equal to those of G, but which has at least two equal invariants. Hence in the study of the number of the characteristic subgroups of G it may be assumed, without loss of generality, that G is of type $(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$, $\alpha_1 > \alpha_2 > \cdots > \alpha_\lambda$. To emphasize the fact that some invariants may be equal to each other the group G will be replaced by the group G'. Let $\alpha_1 - \alpha_2 = \alpha'_1$.

The cyclic subgroup of order p^r, $r < \alpha'_1$, which is generated by each operator of order p^{α_1} contained in G is evidently a characteristic subgroup of G, and G contains no other cyclic characteristic subgroup whenever $p > 2$. When $p = 2$ and $1 < r < \alpha'_1$, G clearly has two and only two cyclic characteristic subgroups of order p^r. Hence the following theorem: *An abelian group of order p^m cannot have more than one characteristic subgroup of order p. A necessary and sufficient condition that such a group G contains at least one characteristic cyclic subgroup of order p^r, p^r being less than $p^{\alpha'_1}$, is that G' has only one largest invariant. A necessary and sufficient condition that G' contains two cyclic characteristic subgroups of order p^r, $1 < r < \alpha'_1$, is that $p = 2$ and that G' has only one largest and also only one next to the largest invariant. No abelian group of order p^m has more than two characteristic cylic subgroups of the same order.*

From the preceding theorem it results that there is a marked difference as regards characteristic subgroups between the groups whose orders are powers of 2 and those whose orders are powers of an odd prime number. Hence we shall assume in what follows, unless the contrary is stated, that $p > 2$. It is easy to prove that every characteristic subgroup of G which involves operators of order p^r must involve the subgroup generated by all the operators of order p^r which are contained in the cyclic subgroups of order p^{α_1} found in G. Hence this characteristic subgroup will be called *the fundamental characteristic subgroup generated by operators of order p^r.* The fundamental characteristic subgroup noted in the first paragraph of the Introduction may therefore also be called, in accord with this more general nomenclature, the fundamental characteristic subgroup generated by operators of order p.

A characteristic subgroup of G cannot involve any operator of order p^{α_1} since all the operators of highest order contained in G are I-conjugate and every abelian group is generated by its operators of highest order. Every

characteristic subgroup of G which involves operators of order p^{a_1-1} must involve the ϕ-subgroup of G since this is composed of all the operators of G which have the property that each of them is the pth power of some other operator of G. Hence the ϕ-subgroup of G is its fundamental characteristic subgroup involving operators of order p^{a_1-1}. When $\lambda > 1$, G has more than one characteristic subgroup which are generated by operators of order p^{a_1-1} These can be arranged linearly so that each contains all those which precede it and involves one more set of I-conjugate operators of order p^{a_1-1} than the one which immediately precedes it.

In fact, the first of these characteristic subgroups is the ϕ-subgroup of G and the remaining $\lambda - 1$ may be obtained by adjoining successively all the operators of smallest order contained in G which are not found in the preceding characteristic subgroup. The complementary subgroups of these characteristic subgroups taken in the reverse order are the λ characteristic subgroups of G which are generated by its operators of order p, and the sum of the numbers of the sets of I-conjugate operators of highest order in each pair formed by one of these characteristic subgroups and its complementary subgroup is $\lambda + 1$. These complementary subgroups are evidently also of complementary types.

All the subgroups of index p under G have the ϕ-subgroup of G for their cross-cut. Hence each such subgroup corresponds to a subgroup of index p in the ϕ-quotient group of G. These subgroups may be divided into λ sets of I-conjugate subgroups corresponding to the λ characteristic subgroups of G which involve only operators of order p. Hence the following theorem: *In any group of order p^m, p being a prime number, all the subgroups of index p which are of the same type are I-conjugate.* It may be noted that it is possible to construct prime power abelian groups in which there are subgroups of every other index which are not I-conjugate. In fact, the abelian group of order p^m and of type $(m - 1, 1)$, $m > 2$, contains cyclic subgroups of every index $> p$ which are not I-conjugate.

A direct proof of the italicized theorem of the preceding paragraph is as follows: The independent generators of G' can be so selected that they differ from the possible independent generators of any given subgroup of index p only as regards one operator, and that the pth of this operator is the remaining independent generator of the subgroup in question. If such a selection of the independent generators of two subgroups of index p and of the same type is made a (1, 1) isomorphism between these subgroups may be established so as to make an independent generator of one of these subgroups correspond to an arbitrary independent generator of the same order in the other. Hence these subgroups correspond in some automorphism of G.*

* Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 73.

It was noted above that the λ characteristic subgroups of G which are generated by operators of order p^{α_1-1} can be arranged linearly so that each includes all those which precede it and that no two of these characteristic subgroups are of the same type. It will be seen that *no abelian group of order p^m, p being an odd prime number, contains two characteristic subgroups which are of the same type,* but it is not always possible to arrange linearly the characteristic subgroups which are generated by operators of order p^r so that each of these subgroups includes all those which precede it.

To obtain all the characteristic subgroups of G which are generated by operators of order p^r we may begin with the fundamental characteristic subgroup K_r of G which is generated by operators of order p^r. In this characteristic subgroup all the operators of order p^r constitute a single set of I-conjugates. When $\lambda > 1$, a characteristic subgroup whose largest operators are of order p^r and which involves two sets of I-conjugate operators of this order can be obtained by adjoining to K_r the smallest set of I-conjugate operators of lowest order found in G but not in K_r. When this lowest order is p there may be more than one set of I-conjugate operators of lowest order found in G which are not contained in K_r. In this case such sets are added successively in order of magnitude beginning with the smallest. We thus obtain a series of characteristic subgroups which can be arranged linearly so that each includes all those which precede it.

A new series of characteristic subgroup may be started by adjoining to K_r the smallest set of I-conjugate operators of lowest order found in G but not contained in the last subgroup of the preceding series. The first subgroup of this new series K'_r involves two or three sets of I-conjugate operators of order p^r according as it does not or does contain more operators of lowest order than K_r. To obtain the various characteristic subgroups of the second series we adjoin to K'_r the smallest set of I-conjugate operators of lowest order found in G but not in K_r. As all the characteristic subgroups of G can be found by continuing this process it has been proved that no two characteristic subgroups of G are of the same type.

It results from this method for finding all the characteristic subgroups of G that while the number of the characteristic subgroups of G cannot exceed the number of the different types of subgroups found in G it may be less than this number. A necessary and sufficient condition that G contains a characteristic subgroup of type $(r_1, r_2, \cdots, r_\lambda)$, where $r_1 \geqq r_2 \geqq \cdots \geqq r_\lambda$ and one or more of the r's may be 0 is that $r_\gamma - r_{\gamma+1} < \alpha_\gamma - \alpha_{\gamma+1}$ $(\gamma = 1, 2, \cdots, \lambda - 1)$ whenever r_γ and $r_{\gamma+1}$ are different from 0. Hence the following theorems: *The number of the characteristic subgroups of any abelian group G' of order p^m, p being an odd prime number, is equal to the number of the characteristic subgroups in a subgroup G of G' which has for*

its independent generators all the different independent generators of G' but has no two independent generators of the same order. If G is of type $(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ then G contains one and only one characteristic subgroup of type $(r_1, r_2, \cdots, r_\lambda)$, $r_1 \geqq r_2 \geqq \cdots \geqq r_\lambda$, where $r_1 < \alpha_1$ and some of the r's may be 0, and $r_\gamma - r_{\gamma+1} \leqq \alpha_\gamma - \alpha_{\gamma+1}$ $(\gamma = 1, 2, \cdots, \lambda - 1)$ whenever r_γ and $r_{\gamma+1}$ are both different from 0.

To illustrate this theorem it may be noted that the abelian group of order p^{10} and of type 3, 3, 2, 1, 1 contains one and only one characteristic subgroup of each of the following types: (1, 1), (1, 1, 1), (1, 1, 1, 1, 1), (2, 2, 1), (2, 2, 1, 1, 1), (2, 2, 2, 1, 1). Hence this group contains six characteristic subgroups besides the identity, and this is also the number of the characteristic subgroups of the abelian group of order p^6 and of type 3, 2, 1 whenever $p > 2$. These characteristic subgroups are of the following types: (1), (1, 1), (1, 1, 1), (2, 1), (2, 1, 1), (2, 2, 1).

It should be noted that every characteristic subgroup of G' has a complementary characteristic subgroup and this is also of the complementary type. When the characteristic subgroups of G' whose largest operator is of order p^r can be arranged linearly so that each includes all those which precede it their complementary characteristic subgroups can be arranged linearly so that each is included in all those which follow it. In particular, the largest characteristic subgroup whose largest operators are of order p^r has for its complementary subgroup the smallest characteristic subgroup whose largest operators are of order $p^{\alpha_1 - r}$, and vice versa.

In view of the reciprocal relations between the characteristic subgroups of G' it may be assumed without loss of generality that $2r \leqq \alpha_1$. From what precedes there results the following theorem: *The number of the different sets of I-conjugate operators of highest order in any characteristic subgroup whose largest operators are of order p^r, increased by the number of the different sets of I-conjugate operators of highest order in its complementary characteristic subgroup, is equal to one more than the number the characteristic subgroups whose largest operator is of order p^r.* In particular, the sum of the numbers of the different sets of I-conjugate operators of highest order in any characteristic subgroup and its complementary characteristic subgroup is independent of the choice of the former characteristic subgroup when the order of the operators of highest order is fixed.

Number of characteristic subgroups when G is of type $(1, 2, 3, \cdots, m)$.

In order to exhibit clearly a method for determining all the characteristic subgroups of any abelian group it seems desirable to consider separately the special case when G is of type $(1, 2, 3, \cdots, m)$ since the formula representing the total number of these subgroups is comparatively simple in this

case. It will be convenient to assume that m may represent an indefinitely large number and to consider separately all the characteristic subgroups, whose operators of highest order is p^r. When r is 1 it is evident that there are m such subgroups and that the orders of these r subgroups are $p, p^2, \cdots, p^m$. The number of sets of I-conjugate operators contained in each of these subgroups is $1, 2, \cdots, m$ respectively. That is, this number is equal to the index of p representing the order of the characteristic subgroup.

When $n = 2$ the smallest characteristic subgroup is of order p^3, since this is the order of the fundamental characteristic subgroup K_2 generated by operators of order p^2. All the operators of order p^2 contained in K_2 are I-conjugate since they are separately powers of operators of order p^m contained in G. This characteristic subgroup involves two of the characteristic subgroups whose operators of highest order are of order p, and if we adjoint to K_2 successively the latter characteristic subgroups which are of orders $p^3, p^4, \cdots, p^m$ respectively there result $m - 1$ characteristic subgroups whose operators of highest order are of order p^2. Each of these $m - 1$ characteristic subgroups has only one independent generator of highest order, and each of these subgroups involves one more complete set of I-conjugate operators of order p^2 than the one which precedes it.

The smallest characteristic subgroup of G which has two independent generators of highest order and involves no operator whose order exceeds p^2 is of order p^5, and involves three of the characteristic subgroups of G' which are separately generated by operators of order p. It involves three sets of I-conjugate operators of order p^2. This characteristic subgroup can be extended by means of characteristic subgroups generated by operators of order p just as K_2 was extended except that the first of these extending subgroups is of order p^4. Each such extension increases by two the number of I-conjugate operators of order p^2, and hence the number of the sets of I-conjugate operators of order p^2 found in the last one of these characteristic subgroups is equal to the total number of the characteristic subgroups continued in G and having no more than two independent of highest order, viz., p^2. This number is $m - 1 + m - 2$.

As this process may be continued until all the characteristic subgroups generated by operators of order p^2 have been found it results that the number of such characteristic subgroups which involve exactly α independent generators of order p^2 is $m - \alpha$, $\alpha = 1, 2, \cdots, m - 1$. The total number of these characteristic subgroups is therefore

$$m(m - 1)/2.$$

The total numbers of the characteristic subgroups generated by the operators of orders p and p^2 contained in G are therefore the sums of the terms,

of the following series of figurate numbers of orders 0 and 1 respectively:

$$1, 1, 1, \cdots \text{ to } m \text{ terms},$$

$$1, 2, 3, \cdots \text{ to } m - 1 \text{ terms}.$$

The fundamental characteristic subgroup K_3 of G generated by operators of order p^3 is of order p^6, and involves three characteristic subgroups generated by operators of order p as well as three such subgroups generated by operators of order p^2. The number of the characteristic subgroups which involve K_3 but have only two independent generators of order p^3 is evidently $m - 2$ and these subgroups involve $1, 2, \cdots, m - 2$ sets of I-conjugate operators of order p^3 respectively. The number of the characteristic subgroups of G which involve only one independent generator of order p^3 but three and only three independent generators of order p^2 is $m - 3$, etc. Hence the number of the characteristic subgroups of G which involve K_3 but have separately only one independent generator of order p^3 is the sum of the series $1, 2, \cdots, m - 2$. Similarly it results that the number of the characteristic subgroups of G which involve K_3 and have separately two and only two independent generators of order p^3 is the sum of the series $1, 2, \cdots, m - 3$.

As this process may be continued until the largest characteristic subgroup of G which is generated by operators of order p^3 has been reached, it results that the number of the characteristic subgroups of G which can be generated by operators of order p^3 is the sum of the figurate numbers of the second order, terminating with $m - 2$. The three special cases considered thus far suggest the following theorem: *The number of the characteristic subgroups of G which are separately generated by operators of order p^r is the sum of the figurate numbers of order $r - 1$, terminating with $m - r + 1$.*

This theorem can easily be proved by mathematical induction since the fundamental characteristic subgroup K_r of G generated by operators of order p^r involves r of the characteristic subgroups generated by operators of order p. Those characteristic subgroups of G which involve only one independent generator of order p^r can be found in the same manner as the characteristic subgroups generated by operators of order p^{r-1} were found with the exception that the series of numbers in the present case consists of $m - r + 1$ numbers while in the preceding case it consisted of $m - r + 2$. The sum of the first $m - r + 1$ figurate numbers of order $r - 2$ which constitute the preceding series is therefore the last term of this series.

Similarly, the sum of the first $m - r$ figurate numbers of order $r - 2$ is the next to the last term in the present series, etc. Hence it results that the present series is composed of figurate numbers if the preceding series was

composed of such numbers, and hence the proof of the theorem in question, is complete. The fact that the number of the characteristic subgroups generated by operators of order p^r is equal to the number of such subgroups generated by operators of order p^{m-r} follows directly from the properties of figurate numbers* as well as from the (1, 1) correspondence between the characteristic subgroups of complementary types.

In the special case under consideration it is easy to see that the number of the characteristic subgroups generated by operators of order p^r is equal to the number of the sets of I-conjugate operators of this order. Hence the latter number is also the sum of the figurate numbers of order $r-1$ when G is of type $(1, 2, 3, \cdots, m)$. As was noted above this result is not affected when G has more than one invariant which is equal to p^α, $1 \leqq \alpha \leqq m$.

* Cf., P. Bachmann, *Niedere Zahlentheorie*, 1910, p. 10.

ERRATA.

Page 152, in the determinants D_1, D_2, and D_3, in place of $h-1$, $i-1$, $2i-h-1$, etc., read a_{h-1}, a_{i-1}, a_{2i-h-1}, etc.

Page 153, the words "the following figures" immediately above the figures refer to the three lower figures. The two upper figures should be three lines higher up, at the beginning of the paragraph; and the inscriptions "Vanishing parallelogram for n odd," and "Vanishing parallelogram for n even," belong to these two upper figures respectively.